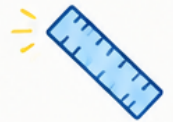




Whole Number Place Value and Expanded Notation Mind Map



Core Concept: The Base-Ten Place Value System

In our base-ten numerical system, the position of each digit determines its actual mathematical value. Each place value column directly to the left represents ten times the value of the place to its right.

- **Units (Ones):** Represents single values from 0 to 9, multiplied by 1.
- **Tens:** Represents sets of ten units, multiplied by 10.
- **Hundreds:** Represents groups of ten tens, multiplied by 100.
- **Thousands:** Represents groups of ten hundreds, multiplied by 1,000.
- **Ten-Thousands:** Represents groups of ten thousands, multiplied by 10,000.



Base-Ten Multiplier Rule

Moving one column left always multiplies the column value by 10: $1 \rightarrow 10 \rightarrow 100 \rightarrow 1,000 \rightarrow 10,000$.

1 Identify Column Alignment

Start at the far-right digit to determine the Units column, then read leftward through Tens, Hundreds, Thousands, and Ten-Thousands.

2 Evaluate Digit Value

Multiply each specific face digit by its corresponding positional multiplier (1, 10, 100, 1,000, or 10,000).

Ten-Thousands	Thousands	Hundreds	Tens	Units
10,000	1,000	100	10	1



Worked Numeric Example: 45,208 in Three Forms

A standard five-digit integer can be expressed in three complementary mathematical notations: standard form, expanded form, and word form. Below is the complete breakdown of 45,208.

- **Standard Form:** 45,208
- **Ten-Thousands Place:** $4 \times 10,000 = 40,000$
- **Thousands Place:** $5 \times 1,000 = 5,000$
- **Hundreds Place:** $2 \times 100 = 200$
- **Tens Place:** $0 \times 10 = 0$
- **Units (Ones) Place:** $8 \times 1 = 8$
- **Expanded Form:** $40,000 + 5,000 + 200 + 8$
- **Word Form:** Forty-five thousand, two hundred eight

i Zero as an Addend

In expanded notation, terms equaling zero ($0 \times 10 = 0$) are typically omitted from the final sum without changing the total value: $40,000 + 5,000 + 200 + 8$.

Place Value & Expanded Notation



Common Mistake Alert: Placeholder Zero vs. Missing Digit

A frequent operational error occurs when learners drop zero from standard form or misalign place values during expanded notation reconstruction.

- **Incorrect Form:** Omitting the zero yields 4,528 (four thousand, five hundred twenty-eight), which is ten times too small in the leading places.
- **Correct Interpretation:** The digit 0 holds the Tens column, ensuring that 5 remains in the Thousands place and 4 remains in the Ten-Thousands place.
- **Expanded Verification:** $40,000 + 5,000 + 200 + 8 = 45,208$, confirming all five positional ranks remain intact.



Common Mistake: Dropping the Placeholder Zero

Never eliminate zero when converting expanded form back into standard form. If a place value has no non-zero value, you must write '0' as a placeholder.



Does the given expanded sum contain an empty place value rank?

A positional rank (e.g., Tens) has no listed addend

Insert a '0' placeholder into that exact column in standard form.

Every rank from highest power of 10 down to ones has a positive addend

Write each digit sequentially according to its place value column.



Multi-Digit Addition and Regrouping Mind Map



Core Addition Hub: Vertical Alignment & Place Value

Multi-digit addition combines two or more addends to find a total sum. Setting up the problem with exact vertical alignment ensures each digit is combined with its matching place value.

- Align numbers vertically strictly by place value: ones over ones, tens over tens, hundreds over hundreds, and thousands over thousands.
- Begin computation exclusively from the rightmost column (ones) and progress leftward.
- Record single digits in the answer space directly beneath the active working column.



Alignment Tip

Always write larger or smaller addends aligned to the right decimal boundary or ones column, never along the left edge.



Estimation and Verification Checks

Verify arithmetic results using rounding to ensure the calculated answer is mathematically reasonable before finalizing.

- Round 3,678 to the nearest thousand (4,000) or nearest hundred (3,700).
- Round 4,855 to the nearest thousand (5,000) or nearest hundred (4,900).
- Estimated benchmark: $4,000 + 5,000 = 9,000$ (or $3,700 + 4,900 = 8,600$).
- Compare the exact answer (8,533) to the estimated benchmark (8,600) to confirm reasonableness.



Verification Rule

If your exact sum differs drastically from your rounded estimate, recheck your carried regrouping numbers across every column.

Practice Problem:
 $2,749 + 5,683$
(Show Carried Digits)

Estimation Benchmark



Step-by-Step Regrouping: $3,678 + 4,855$

When the sum in any column reaches 10 or greater, regroup the tens portion to the top of the adjacent column to the left.

- Add ones: $8 + 5 = 13$. Write 3 in the ones answer position; carry 1 ten to the top of the tens column.
- Add tens: $1 \text{ (carried)} + 7 + 5 = 13$. Write 3 in the tens answer position; carry 1 hundred to the top of the hundreds column.
- Add hundreds: $1 \text{ (carried)} + 6 + 8 = 15$. Write 5 in the hundreds answer position; carry 1 thousand to the top of the thousands column.
- Add thousands: $1 \text{ (carried)} + 3 + 4 = 8$. Write 8 in the thousands answer position. Final calculated sum is 8,533.



Vertical Calculation

	Th	H	T	O
		+1	+1	+1
	3	6	7	8
+	4	8	5	5
	8	5	3	3



Common Mistake: Dropped Carried Values

Do not write two digits into a single column's answer box (such as writing '13' in the ones column). Forgetting to add the carried digit from the previous column produces an incorrect total like 7,423 or 8,423.

Multi-Digit Addition & Regrouping



Decision: When to Regroup

Does the sum of digits in the active column equal 10 or greater?

Column sum is
between 0 and 9

Write the sum directly
in the answer space
beneath that column.
Carry 0 to the next
column.

Column sum is
10 or greater

Write only the ones
digit of the sum in the
answer row. Carry the
tens digit to the top of
the next left column.



Step-by-Step Example

1

Step 1: Column Alignment

Stack 3,678 and 4,855 vertically with a horizontal line beneath and a plus sign (+) to the left.

2

Step 2: Ones Column Addition

Calculate $8 + 5 = 13$. Write down 3 in the ones place; place small digit 1 above the 7 in the tens place.

3

Step 3: Tens Column Addition

Calculate $1 + 7 + 5 = 13$. Write down 3 in the tens place; place small digit 1 above the 6 in the hundreds place.

4

Step 4: Hundreds Column Addition

Calculate $1 + 6 + 8 = 15$. Write down 5 in the hundreds place; place small digit 1 above the 3 in the thousands place.

5

Step 5: Thousands Column Addition

Calculate $1 + 3 + 4 + 8$. Write down 8 in the thousands place to complete the total sum: 8,533.



Common Mistake

Do not write two digits into a single column's answer box (such as writing '13' in the ones column). Forgetting to add the carried digit from the previous column produces an incorrect total like 7,423 or 8,423.



Multi-Digit Subtraction and Decomposition Across Zeros



Core Subtraction Anatomy

Subtraction determines the difference between two quantities by evaluating how much remains when a subtrahend is removed from a minuend. In base-ten arithmetic, place values align vertically from right to left, starting with units.

- **Minuend:** The initial quantity from which another value is subtracted (top number in column alignment).
- **Subtrahend:** The quantity being subtracted from the minuend (bottom number in column alignment).
- **Difference:** The resulting amount after completing the subtraction across all place-value columns.
- **Relationship Identity:** Minuend - Subtrahend = Difference, verifiable via Difference + Subtrahend = Minuend.



Terminology Verification

In vertical subtraction, always subtract the subtrahend digit from the minuend digit within each column. Swapping their positions changes the mathematical value.

Multi-Digit Subtraction & Decomposition Across Zeros



Worked Example: 5,002 Minus 2,347

Step-by-step arithmetic decomposition demonstrating sequential borrowing across consecutive zero digits in the hundreds and tens positions.

- **Units Column:** 2 is less than 7. Regrouping required, but tens and hundreds columns are both 0.
- **Thousands Decomposition:** 5 thousands becomes 4 thousands; transfers 10 hundreds to the hundreds place.
- **Hundreds Decomposition:** 10 hundreds decomposes to 9 hundreds; transfers 10 tens to the tens place.
- **Tens Decomposition:** 10 tens decomposes to 9 tens; transfers 10 units to the units place, yielding 12 units.
- **Final Column Calculations:** Units: $12 - 7 = 5$; Tens: $9 - 4 = 5$; Hundreds: $9 - 3 = 6$; Thousands: $4 - 2 = 2$.
- **Computed Difference:** 2,655. Verification Check: $2,655 + 2,347 = 5,002$.



Verification Check

Always add the resulting difference to the subtrahend to confirm the sum matches the original minuend:
 $2,655 + 2,347 = 5,002$.

	Thousands	Hundreds	Tens	Units
	5 4	0 9	0 9	2 12
-	2	3	4	7
	2	6	5	5



Sequential Regrouping and Borrowing Mechanism

When a minuend digit in any place value is smaller than the corresponding subtrahend digit, standard decomposition must be performed. When adjacent place values contain zeros, regrouping cascades from the nearest non-zero place value on the left.

- **Base-Ten Trading:** One unit of a higher place value decomposes into 10 units of the immediately adjacent lower place value.
- **Cascading Trade:** A zero cannot directly lend; it must first receive value from the left before decomposing to supply the next right digit.
- **Intermediate Decomposition:** Each zero in the cascade chain becomes 10 upon receiving value, and then decomposes to 9 as it lends 1 to the next place.



Avoid Inversion Error

Common Mistake: Subtracting in reverse (smaller top digit minus larger bottom digit, such as writing $7 - 2 = 5$ when evaluating $2 - 7$). Always decompose from the left column when the minuend digit is smaller than the subtrahend digit.

Steps

- 1 **Locate Non-Zero Column**
Scan leftward from the deficit column across any intermediate zero digits until reaching the first non-zero place value.
- 2 **Decompose the Non-Zero Digit**
Reduce the non-zero digit by 1 and transfer 10 units to the adjacent zero column immediately to its right.
- 3 **Cascade Through Intermediate Zeros**
Decompose the resulting 10 into 9, transferring 10 to the next column rightward; repeat until reaching the target column.
- 4 **Execute Column Subtraction**
Combine the transferred 10 with the existing minuend digit, then subtract the subtrahend digit directly.



Common mistake

-8 is less than -3, because it is further to the left on the number line.



Multiplication Concepts, Arrays, and Fact Families Mind Map



Multiplication Core Concepts

Multiplication represents equal-sized groups combined together. Instead of repeatedly adding the same number, multiplication consolidates equal quantities into an efficient single calculation of factors producing a product.

- Factor $\times$ Factor = Product
- Equal groups: 6 groups of 7 items yield 42 total items
- Repeated addition equivalent: $7 + 7 + 7 + 7 + 7 + 7 = 42$

Multiplication Identity Rule

Any factor multiplied by 1 equals itself ($a \times 1 = a$), and any factor multiplied by 0 equals 0 ($a \times 0 = 0$).

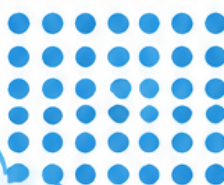


Rectangular Grid Arrays & Commutativity

An array organizes items into uniform rows and columns. Rotating an array 90 degrees demonstrates the commutative property of multiplication, proving that changing the order of the factors does not change the resulting product.

- Array model: 6 rows of 7 counters equals 42 counters
- Rotated orientation: 7 rows of 6 counters also equals 42 counters
- Commutative property statement: $a \cdot b = b \cdot a$, meaning $6 \times 7 = 7 \times 6 = 42$

6 rows $\times$ 7 counters
= 42 counters



7 rows $\times$ 6 counters
= 42 counters



Multiplication & Fact Families



Fact Family Relations & Inverse Operations

Division is the direct inverse of multiplication. A complete fact family uses three numbers to establish two multiplication equations and two corresponding division equations.

- Primary multiplication: $6 \times 7 = 42$
- Commutative multiplication: $7 \times 6 = 42$
- Primary division: $42 \div 6 = 7$
- Inverse division: $42 \div 7 = 6$



Common Mistake Alert

Do not confuse multiplication with addition: $6 \times 7 = 42$, whereas $6 + 7 = 13$. Multiplication combines repeated equal sets, while addition combines individual unequal or single addends.

Steps to Remember

- 1 **Identify Given Numbers**
Note the three target values: factors 6 and 7, and product 42.
- 2 **State Both Products**
Calculate $6 \times 7 = 42$, then reverse the factors to confirm $7 \times 6 = 42$.
- 3 **Divide by First Factor**
Divide the total product by the first factor: $42 \div 6 = 7$.
- 4 **Divide by Second Factor**
Divide the total product by the second factor: $42 \div 7 = 6$ to complete the four-part family.



Commutative Property

The commutative property states that the order of the factors does not change the product.

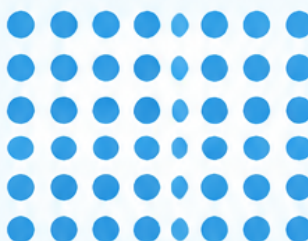
$$6 \times 7 = 42$$

$$7 \times 6 = 42$$

$$42 \div 6 = 7$$

$$42 \div 7 = 6$$

Visual Check



6 rows
 $\times$ 7 columns
= 42

Long Multiplication with Multi-Digit Factors — Radial Mind Map



Core Concept: Partial Products Architecture

Multi-digit multiplication breaks a complex factor into expanded positional parts. When multiplying a multi-digit top factor by a multi-digit bottom factor, you multiply the top number separately by each place-value digit of the multiplier. The resulting rows are partial products that must be added column-wise to find the final product.

- Align factors vertically by place value before multiplying.
- Multiply the top factor across all places by the ones digit first.
- Insert a zero place-holder in the ones column before multiplying by the tens digit.
- Add all intermediate partial product rows using aligned column addition.



Zero Place-Holder Rule

When multiplying by a tens digit, the actual value is ten times larger. A zero must occupy the ones place of that row before calculating individual products.

1

Calculate First Partial Product (Ones Digit)

Multiply 248 by the 6 in the ones place of 36.
 $8 \times 6 = 48$ (write 8, carry 4); $4 \times 6 = 24$
plus $4 = 28$ (write 8, carry 2); $2 \times 6 = 12$
plus $2 = 14$. The first partial product is exactly 1,488.

2

Place Holder and Multiply Tens Digit

Insert 0 in the ones place of the second partial row.
Multiply 248 by 3 tens (30). $8 \times 3 = 24$ (write 4, carry 2);
 $4 \times 3 = 12$ plus $2 = 14$ (write 4, carry 1); $2 \times 3 = 6$ plus $1 = 7$.
The second partial product is 7,440.

3

Sum Aligned Partial Products

Add 1,488 and 7,440 column by column:
 $8 + 0 = 8$; $8 + 4 = 12$ (write 2, carry 1); $1 + 4 + 9 = 14$
 $1 + 7 = 8$. The final product is 8,928.

Multi-Digit Multiplication:
 248×36



Worked Example: 248×36 Step-by-Step

Follow the structured mathematical layout below to verify each stage of the standard multi-digit algorithm.

- Top factor (multiplicand): 248
- Bottom factor (multiplier): 36 (3 tens + 6 ones)
- Row 1 calculation: $248 \times 6 = 1,488$
- Row 2 calculation: $248 \times 30 = 7,440$
- Combined sum: $1,488 + 7,440 = 8,928$



Common Mistake Alert

Dropping the trailing zero in the tens-digit partial product causes 744 to be added directly to 1,488, yielding an erroneous sum of 2,232 instead of the correct 8,928. Always verify place-holder alignment.



Does the multiplier have additional digits to the left?

Yes (hundreds place or higher)

Append two trailing zero place-holders (00) for the hundreds row before multiplying the digits, then continue row by row.

No (tens place was the final digit)

Draw an addition boundary line beneath the partial products and perform standard column-wise addition from right to left.



Student & Tutor Verification Checklist

Review your written computation against these core structural checkpoints before finalizing your solution.

- Are all single-digit carried values written cleanly above the respective column?
- Did you cross out or erase carried numbers from the ones step before starting the tens step?
- Is exactly one zero place-holder visible at the right end of the second partial product?
- Are all decimal and whole-number columns aligned precisely over one another?



Tutor / Parent Verification Note



Student Self-Check Score (1 to 5)



Division Fundamentals and Equal Sharing Models



Core Division Terminology & Anatomy

Division decomposes a total quantity into equal groups or measures. Every division statement relates four core components:



Dividend $\div$ Divisor = Quotient (with Remainder).

- **Dividend:** The total quantity being split into parts (e.g., 29 items).
- **Divisor:** The number of equal groups or group size (e.g., 4 groups).
- **Quotient:** The resulting whole number of items per group (e.g., 7 items).
- **Remainder:** The leftover amount strictly smaller than the divisor (e.g., 1 remaining).



Undefined Operation: Division by Zero

Division by zero ($a \div 0$) is undefined. A quantity cannot be distributed into zero groups or measured by zero-sized parts.

1

Steps for Division

1

Identify Total and Group Count

Assign the total count to the dividend and the intended partition count to the divisor.

2

Perform Equal Distribution

Distribute items equally to determine the maximum whole quotient per group.

3

Isolate Leftovers

Calculate the remainder:
 $\text{Remainder} = \text{Dividend} - (\text{Divisor} \times \text{Quotient})$.



Is the remaining quantity after distribution zero?

Remainder equals zero

The dividend is evenly divisible by the divisor; write the final quotient as an exact integer.

Remainder is greater than zero

Express the result with remainder notation (Quotient R[Remainder]) or as a mixed number (Quotient Remainder/Divisor).

Division & Equal Sharing



Common Mistakes

- Confusing the dividend and divisor (e.g., writing $4 \div 29$ instead of $29 \div 4$).
- Forgetting to include the remainder (e.g., writing $29 \div 4 = 7$ instead of $7 \text{ R}1$).
- Using a remainder that is equal to or greater than the divisor (e.g., $10 \text{ R}4$ with divisor 4).



Division by Zero Alert

Division by zero ($a \div 0$) is undefined. Do not attempt to divide by zero.



Worked Sharing Model & Arithmetic Verification

A full numeric walk-through demonstrating partition sharing and verification via inverse operations.

- **Expression:** $29 \div 4$
- **Partitioning:** 29 items distributed equally into 4 distinct groups yield 7 items per group with 1 item remaining.
- **Standard Notation:** $29 \div 4 = 7 \text{ R}1$
- **Inverse Check Formula:** $\text{Dividend} = (\text{Divisor} \times \text{Quotient}) + \text{Remainder}$
- **Direct Calculation:** $(4 \times 7) + 1 = 28 + 1 = 29$



Remainder Validity Rule

A remainder must always be strictly less than the divisor ($0 \leq R < \text{Divisor}$). If $R \geq \text{Divisor}$, increment the quotient.



Sharing Model: $29 \div 4$



7 items



7 items



7 items



7 items



1 item left over

$$4 \times 7 + 1 = 29$$

1 Step 1: Divide

Determine how many whole times 4 fits into 29 ($4 \times 7 = 28$, $4 \times 8 = 32$; select 7).

2 Step 2: Subtract to Find Remainder

Subtract the product from the dividend:
 $29 - 28 = 1$.

3 Step 3: Verify with Multiplication

Multiply quotient by divisor and add remainder:
 $4 \times 7 + 1 = 29$.



Long Division Algorithm Mind Map



Core Process: The Four-Step Division Loop

Long division breaks multi-digit division into four repeating actions performed place by place from left to right. Once a digit is processed, bring down the next place value and repeat the sequence until no digits remain in the dividend.

- **Divide:** Determine how many times the divisor fits into the active working value without exceeding it.
- **Multiply:** Multiply the divisor by the newly written quotient digit.
- **Subtract:** Subtract that product from the active working value to find the remainder.
- **Bring Down:** Bring down the adjacent digit of the dividend to form the next active working value.



Iterative Loop Rule

Every time you bring down a digit, you **MUST** execute a division step and place a digit into the quotient above that place value, even if the value is 0.

1

Step-by-Step Loop

1

Step 1: Divide Hundreds

Look at the first digit of 754. Divide 7 by 5. 5 goes into 7 exactly 1 time. Write 1 in the quotient directly over the 7.

2

Step 2: Multiply & Subtract Hundreds

Multiply 1 by 5 = 5. Subtract 5 from 7 to get 2.

3

Step 3: Bring Down Tens & Divide

Bring down 5 to make 25. Divide 25 by 5 = 5. Write 5 in the quotient above the 5. Multiply 5 by 5 = 25. Subtract 25 from 25 to get 0.

4

Step 4: Bring Down Units, Record Zero & Remainder

Bring down 4 to form 04. Divide 4 by 5: 5 goes into 4 exactly 0 times. Write 0 in the quotient above the 4. Multiply 0 by 5 = 0. Subtract 0 from 4 to leave a remainder of 4. Final result: 150 R4



Worked Demonstration: 754 Divided by 5

Follow the structured tabular alignment below to verify each arithmetic operation through every place-value column.

- **Hundreds Column:** 7 divided by 5 yields quotient digit 1, product 5, remainder 2.
- **Tens Column:** Bring down 5 to form 25. 25 divided by 5 yields quotient digit 5, product 25, remainder 0.
- **Units Column:** Bring down 4 to form 04. 4 divided by 5 yields quotient digit 0, product 0, remainder 4.
- **Verification Equation:** (Divisor $\times$ Quotient) + Remainder = $(5 \times 150) + 4 = 750 + 4 = 754$.



Common Mistake: The Forgotten Zero

A frequent error is stopping when 4 cannot be divided by 5, recording an answer of 15 R4. Because the digit 4 was brought down, a division step must be completed: 4 divided by 5 equals 0 with remainder 4. Always record the 0 in the quotient units place to maintain proper place value. Check: $15 \times 5 = 75$ (incorrect), whereas $150 \times 5 = 750$ (correct).



Verification Formula

$$(5 \times 150) + 4 = 754$$

Long Division Algorithm: Divide, Multiply, Subtract, Bring Down



Self-Check Decision Matrix

Use this systematic diagnostic check immediately after completing any long division problem.

- Check remainder magnitude against the divisor at every subtraction stage.
- Verify quotient length matches the total number of processed dividend places.

Is the subtraction remainder strictly less than the divisor, and has every brought-down digit received a quotient digit?

Yes, remainder < divisor, and all columns have quotient entries



Proceed to the verification formula: Multiply quotient by divisor, add remainder, and verify it matches the dividend exactly.

No, remainder is greater than or equal to divisor



Increase the quotient digit for that specific column by at least 1 and re-calculate the subtraction.

No, a digit was brought down but no quotient digit was placed



Place a 0 in the quotient directly over that brought-down place value before concluding or bringing down another digit.



Fully Worked Example: $754 \div 5$

$$\begin{array}{r} 150 \text{ R}4 \\ 5 \overline{) 754} \\ \underline{- 5} \\ 25 \\ \underline{- 25} \\ 04 \\ \underline{- 0} \\ 4 \end{array}$$



Order of Operations: PEMDAS and Parenthetical Grouping



PEMDAS Hierarchy & Rules

The order of operations guarantees a unique, correct value for every mathematical expression. Operations are resolved in strictly tiered ranks, where grouping symbols dictate priority over arithmetic operators.

- 1 Parentheses & Grouping:**
Resolve nested terms from innermost to outermost (parentheses, brackets, braces, fraction bars).
- 2 Exponents & Roots:**
Evaluate all powers, squared values, and radicals from left to right.
- 3 Multiplication & Division:**
Possess identical precedence; evaluate strictly in order of appearance from left to right.
- 4 Addition & Subtraction:**
Possess identical precedence; evaluate strictly in order of appearance from left to right.



Equal Priority Rule

Multiplication does NOT automatically outrank division, nor does addition outrank subtraction. When operations share equal priority, always evaluate them strictly from left to right.

Order of Operations (PEMDAS)



Step-by-Step Worked Evaluation

Evaluate the arithmetic expression:

$$8 + (3 \times (4 + 2) - 6) / 2$$

- Target Expression: $8 + (3 \times (4 + 2) - 6) / 2$
- Final Result: 14

- 1 Innermost Parentheses**
Evaluate the innermost grouped expression $(4 + 2)$ to get 6. The expression becomes: $8 + (3 \times 6 - 6) / 2$.
- 2 Multiply Within Outer Parentheses**
Inside the remaining parentheses $(3 \times 6 - 6)$, multiplication takes precedence over subtraction: $3 \times 6 = 18$. The expression becomes: $8 + (18 - 6) / 2$.
- 3 Complete Parenthetical Evaluation**
Complete the remaining operation inside parentheses: $18 - 6 = 12$. The expression becomes: $8 + 12 / 2$.
- 4 Division Precedence**
Division takes precedence over addition: compute $12 / 2 = 6$. The expression simplifies to: $8 + 6$.
- 5 Final Addition**
Perform the final addition: $8 + 6 = 14$.



Common Mistake Alert & Priority Pitfalls

A frequent misconception among students is performing addition before subtraction simply because the letter 'A' precedes 'S' in the acronym PEMDAS. Addition and subtraction share equal priority and must be solved in sequence from left to right.

- **Incorrect Evaluation:** In $10 - 3 + 2$, computing $(3 + 2 = 5)$ first leads to $10 - 5 = 5$ (Incorrect).
- **Correct Evaluation:** Evaluating left to right yields $10 - 3 = 7$, then $7 + 2 = 9$ (Correct).
- **Same logic applies to division and multiplication:** In $12 / 3 \times 2$, compute $12 / 3 = 4$ first, then $4 \times 2 = 8$, not $12 / 6 = 2$.



Warning: Avoid Premature Addition

Never calculate addition before subtraction when subtraction appears earlier in the expression from left to right. Equal-tier operations must always be evaluated in the exact order they appear.

Unit 9: Divisibility Rules Mind Map



Core Divisibility Rules at a Glance

Divisibility tests allow rapid determination of whether an integer divides evenly into another without performing long division. Rules separate primarily into terminal digit checks (last digit or digits) and digit sum tests.

- Rule for 2: The last digit is even (0, 2, 4, 6, 8).
- Rule for 3: The sum of all digits is divisible by 3.
- Rule for 4: The number formed by the last two digits is divisible by 4.
- Rule for 5: The last digit is 0 or 5.
- Rule for 6: The number satisfies both Rule 2 (even) and Rule 3 (digit sum divisible by 3).
- Rule for 9: The sum of all digits is divisible by 9.
- Rule for 10: The terminal digit is exactly 0.

2

Rule for 2, 5, 10 (Last Digit Tests)

A number is divisible by:

- 2 if the last digit is even (0, 2, 4, 6, 8).
- 5 if the last digit is 0 or 5.
- 10 if the last digit is 0.

Examples:

2, 5, 10 → 32, 75, 120

3

Rule for 3 & 9 (Digit Sum Tests)

A number is divisible by:

- 3 if the sum of all digits is divisible by 3.
- 9 if the sum of all digits is divisible by 9.

Examples:

3, 9 → 18, 27, 324, 999

4

Rule for 4 (Last 2 Digits)

A number is divisible by 4 if the number formed by the last two digits is divisible by 4.

Examples:

4 → 16, 28, 432, 1,204

5

Rule for 6 (Combined 2 & 3)

A number is divisible by 6 if it is divisible by both 2 and 3.

Examples:

6 → 12, 18, 30, 126

6

Rule for 10

A number is divisible by 10 if the terminal digit is exactly 0.

Examples:

10 → 20, 50, 130, 2,460

DIVISIBILITY RULES



Worked Multi-Factor Verification: Testing 1,620

Verification of the composite integer 1,620 across all primary divisibility rules demonstrates how terminal checks and digit sums operate concurrently.

- Test for 2: The terminal digit is 0 (even), so 1,620 is divisible by 2 ($1,620 \div 2 = 810$).
- Test for 3: Digit sum is $1 + 6 + 2 + 0 = 9$. Since 9 is divisible by 3, 1,620 is divisible by 3 ($1,620 \div 3 = 540$).
- Test for 4: The last two digits form 20. Since $20 \div 4 = 5$, 1,620 is divisible by 4 ($1,620 \div 4 = 405$).
- Test for 5: The terminal digit is 0, so 1,620 is divisible by 5 ($1,620 \div 5 = 324$).
- Test for 6: Because 1,620 is divisible by both 2 and 3, it is divisible by 6 ($1,620 \div 6 = 270$).
- Test for 9: Digit sum is $1 + 6 + 2 + 0 = 9$. Since 9 is divisible by 9, 1,620 is divisible by 9 ($1,620 \div 9 = 180$).
- Test for 10: The terminal digit is 0, so 1,620 is divisible by 10 ($1,620 \div 10 = 162$).



Composite Rule Note

For composite divisors like 6 (where $6 = 2 \times 3$ and $\gcd(2, 3) = 1$), satisfying both coprime factors guarantees divisibility by the composite base.



Common Mistake: Odd Terminal Digits and Divisibility by 3

Never assume that odd terminal digits fail divisibility by 3, or that an odd terminal digit automatically guarantees divisibility by 3.

Divisibility by 3 depends entirely on the sum of all digits, not the final digit alone.

For instance, 21 and 27 end in odd digits and are divisible by 3, while 25 ends in an odd digit and is not.



Practice Verification Card

Apply the radial branches to test divisibility for the student practice number provided below.

- Record results for each branch systematically.
- State whether the digit sum test or terminal digit test was utilized.

Practice Number: 4,356 — Divisible by 2, 3, 4, 6, 9? Show digit sum and terminal checks:

Practice Number: 7,815 — Divisible by 3, 5, 10? Show checks and explain terminal digit failure for 10:



Prime Numbers, Composite Numbers, and Sieve Verification Mind Map



Core Taxonomy: Number Classification System

Every whole number strictly greater than 0 falls into one of three rigorous categories based solely on its distinct positive factors.

- **Unique Unit (1):** Exactly 1 positive factor (1 itself). It is fundamentally excluded from primes and composites.
- **Prime Numbers:** Integers greater than 1 with exactly two distinct positive factors: 1 and the number itself.
- **Composite Numbers:** Integers greater than 1 with three or more distinct positive factors, allowing non-trivial factorization.



Mathematical Axiom: Why 1 Is Neither Prime Nor Composite

By mathematical definition, a prime number must have exactly two distinct positive factors. The number 1 has only one positive factor (1). If 1 were defined as prime, the Fundamental Theorem of Arithmetic (unique prime factorization) would fail, as numbers could have infinite factor representations (e.g., $6 = 2 * 3 = 1 * 2 * 3 = 1 * 1 * 2 * 3$).

Natural Number Taxonomy



Worked Numeric Proof: Factors of 37 versus 39

Examine the exhaustive factor pairs to classify numbers definitively as prime or composite.

Test Number 37:

- Factor pairs tested up to $\sqrt{37}$ (approx 6.08).
- Divisibility checked against 2, 3, 5: no integer quotients.
- Factors of 37: exactly {1, 37}.
- **Conclusion: 37 is Prime.**

Test Number 39:

- Factor pairs tested: $1 * 39 = 39$, $3 * 13 = 39$.
- Factors of 39: {1, 3, 13, 39}.
- **Conclusion: 39 is Composite** because it possesses 4 distinct factors.

Steps for Classification

- Step 1: Calculate the Integer Square Root Bound**
For integer n , find the largest integer k such that $k^2 \leq n$. Only prime divisors less than or equal to k need testing.
- Step 2: Systematic Divisibility Sweep**
Test division of n by consecutive primes 2, 3, 5, ... up to k . If any divide evenly with remainder 0, list the factor pair and terminate: n is composite.
- Step 3: Conclude Classification**
If no primes $\leq k$ divide n without remainder, n has exactly two factors (1 and n) and is rigorously certified prime.

Decision Flow

Does the integer $n > 1$ have any divisor other than 1 and n ?

Has exactly two factors
(1 and itself)

Classify as Prime
(e.g., 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37).

Has three or more factors

Classify as Composite
(e.g., 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 39).



Sieve of Eratosthenes: Systematic Verification Grid

The Sieve of Eratosthenes isolates primes by iteratively marking multiples of each discovered prime starting from 2.

- 1 is struck out immediately as a unit.
- Circle 2 (first prime); eliminate all higher multiples of 2 (4, 6, 8, ...).
- Circle 3 (next uncrossed number); eliminate remaining multiples of 3 (9, 15, 21, ...).
- Proceed to 5 and 7; all remaining unmarked integers under 100 are verified prime numbers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



Common Mistake Alert

Odd ≠ Prime

Never assume all odd numbers are prime. While all primes greater than 2 are odd, many odd numbers are composite (e.g., $9 = 3 * 3$, $15 = 3 * 5$, $21 = 3 * 7$, $25 = 5 * 5$, $27 = 3 * 9$, $39 = 3 * 13$). Always verify factors.



Prime Factorization & Factor Tree Decomposition Mind Map



Core Concept: Prime Factorization

Every composite number can be broken down into a unique set of prime factors, known as its prime factorization. A prime number has exactly two distinct positive divisors: 1 and itself (such as 2, 3, 5, 7, 11). Two primary visual methods are used to determine this decomposition: branching factor trees and continuous division.

- **Fundamental Theorem of Arithmetic** guarantees uniqueness of prime factors regardless of initial split.
- **Terminal nodes** of every factor tree must be prime numbers.
- Final decomposed form is typically expressed using exponential notation in ascending prime order.



Prime Check Reminder

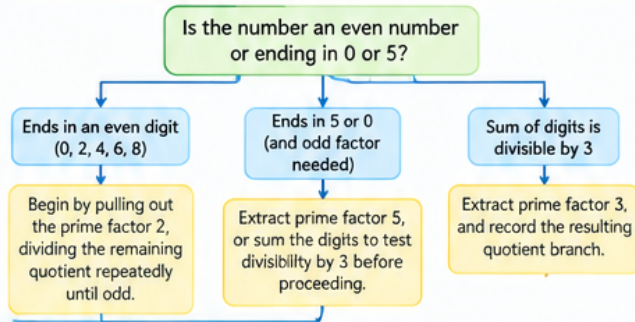
The number 1 is neither prime nor composite. The smallest prime number and only even prime is 2.



Visual Decomposition Methods

Select an appropriate decomposition technique based on the composite value and visual preference.

- **Factor Tree:** Split the starting value into any two factors, repeating down each branch until all tips are prime.
- **Continuous Division (L-Method):** Divide sequentially by the smallest possible prime factor until the final quotient reaches 1.



Prime Factorization & Factor Tree Decomposition



Common Mistakes & Pitfalls

Key conceptual errors to avoid during prime tree decomposition and factorization writing.

- **Premature Termination:** Stopping a branch at a composite number (such as leaving 9, 6, or 4 unexpanded).
- **Including Factor 1:** Writing 1 in the prime factorization string (e.g., $1 \times 2^2 \times 3^2 \times 5$). Prime numbers must be strictly greater than 1.
- **Dropping Intermediate Leaves:** Failing to collect a prime leaf from an earlier shallow branch when assembling the final equation product.



Critical Pitfall: Non-Prime Termination

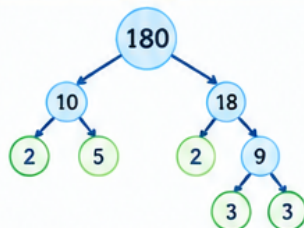
Never stop branching at composite numbers like 4, 6, 8, 9, or 10. A factor tree branch only terminates when its leaf node is strictly prime.



Worked Numeric Example: Factoring 180

Complete decomposition of composite number 180 into its prime factors using tree branching and exponential consolidation.

- Initial split: $180 = 10 \times 18$
- Left branch split: $10 = 2 \times 5$ (both prime, terminal nodes)
- Right branch split: $18 = 2 \times 9$ (2 is prime, terminal node)
- Sub-branch split: $9 = 3 \times 3$ (both prime, terminal nodes)
- All terminal leaf nodes: 2, 5, 2, 3, 3
- Group and sort in ascending order: $2 \times 2 \times 3 \times 3 \times 5$
- Final exponential form: $2^2 \times 3^2 \times 5$



$$2^2 \times 3^2 \times 5$$

- 1 **Step 1: Choose Any Initial Factor Pair**
Divide 180 into two non-trivial factors, such as 10 and 18 (or 2 and 90, or 4 and 45). Draw two downward branches.
- 3 **Step 2: Decompose Composite Branch Nodes**
Split 10 into 2×5 . Both are prime, so circle them as terminal leaves. Split 18 into 2×9 . Circle 2 as terminal.
- 3 **Step 3: Continue to Prime Termination**
Split composite 9 into 3×3 . Both 3s are prime. Circle both leaves. No composite branches remain.
- 4 **Step 4: Verify Product and Write Exponential Form.**
Multiply all circled leaves: $2 \times 2 \times 3 \times 3 \times 5 = 180$. Write in prime-power form: $2^2 \times 3^2 \times 5$.



Guided Practice & Factor Verification

Complete the factor decomposition steps for the assigned target values, circling each terminal prime node.

- Ensure all branches end in circled prime numbers.
- Express each final answer in exponential prime form.

Practice Number: Decompose 72 into prime factor tree nodes

Write 72 in ascending exponential prime form

Practice Number: Decompose 250 into prime factor tree nodes

Write 250 in ascending exponential prime form





Greatest Common Factor (GCF) Determination



Core Concept: Greatest Common Factor (GCF)

The Greatest Common Factor (GCF), also known as the greatest common divisor, is the largest positive integer that divides evenly into two or more given numbers without leaving a remainder.

- A factor divides a number completely with zero remainder.
- Common factors are values shared across the factor lists of multiple numbers.
- The GCF is the maximum element among all shared common factors.
- GCF is essential for reducing fractions to simplest form and factoring algebraic expressions.



Core Definition

For integers a and b , $GCF(a, b)$ is the largest integer d such that d divides a and d divides b .



Method 1: Listing All Factors

Systematically list every positive factor of each target number in ascending order, identify every number present in both lists, and select the greatest value.

- Factors of 48: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.
- Factors of 72: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.
- Shared common factors: {1, 2, 3, 4, 6, 8, 12, 24}.
- Largest shared value: 24.

Steps

- 1 List Factors of First Number
Divide 48 by integers starting at 1 to record all factor pairs: 1x48, 2x24, 3x16, 4x12, 6x8.
- 2 List Factors of Second Number
Divide 72 by integers starting at 1 to record all factor pairs: 1x72, 2x36, 3x24, 4x18, 6x12, 8x9.
- 3 Intersect Lists
Extract all matching values occurring in both lists: 1, 2, 3, 4, 6, 8, 12, and 24.
- 4 Select the Maximum
The largest value in the intersection is 24, confirming $GCF(48, 72) = 24$.



Method 2: Prime Factorization and Venn Overlap

Deconstruct both numbers into their prime factor trees. Represent the prime factors within a Venn diagram where the overlapping center captures only shared prime instances.

- Prime factorization of 48: $2 \times 2 \times 2 \times 2 \times 3 = 2^4 \times 3^1$.
- Prime factorization of 72: $2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$.
- Shared prime factors in intersection: three 2s and one 3 ($2^3 \times 3^1$).
- Calculated product of overlap: $2 \times 2 \times 2 \times 3 = 24$.



Common Mistake Warning

Multiplying all prime factors from both circles produces the Least Common Multiple ($LCM = 2^4 \times 3^2 = 144$), not the GCF. The GCF requires multiplying ONLY the primes present in the central shared intersection ($2^3 \times 3 = 24$).

Greatest Common Factor (GCF)



Method 3: Euclidean Subtraction and Division

The Euclidean algorithm leverages the property that $GCF(a, b) = GCF(b, a - b)$ when $a > b$, providing a fast iterative method without full factorization.

- Start with pair (72, 48).
- Subtract smaller from larger: $72 - 48 = 24$. New pair: (48, 24).
- Subtract again: $48 - 24 = 24$. New pair: (24, 24).
- When numbers match, the remaining value is the GCF: 24.



Which method is best suited for your problem?

1

Numbers are small (under 50)



Use Method 1 (listing factors) for quick visual inspection.

2

Medium composite numbers with clear factor trees



Use Method 2 (prime factorization and Venn diagram overlap) to isolate shared powers.

3

Large numbers with difficult prime trees



Use Method 3 (Euclidean subtraction or division) to reduce numbers systematically.

Least Common Multiple (LCM) Reference Card



Core Definition & Fundamental Concept

The Least Common Multiple (LCM) of two or more non-zero integers is the smallest positive integer that is divisible by all numbers in the set without leaving a remainder. In fractional arithmetic, the LCM of denominators serves as the Lowest Common Denominator (LCD).

- **Multiple:** The product of a given number multiplied by any positive integer (e.g., multiples of 4 are 4, 8, 12, 16...).
- **Common Multiple:** A shared multiple that appears in the multiple lists of two or more distinct numbers.
- **Least Common Multiple:** The lowest positive value among all shared common multiples.



Terminology Clarification

Multiples grow larger or stay equal to the base numbers. Factors divide evenly into the base numbers and are smaller or equal.



Method 1: Multiple Listing Strategy

The listing method identifies the LCM by writing out successive positive multiples for each integer until the first matching value is observed across all lists.

- Best suited for smaller values such as single-digit integers.
- Requires systematically generating products in ascending numerical order.

1 Step 1: List Multiples of the First Integer

Write consecutive positive multiples of 12:
12, 24, 36, 48, 60, 72...

2 Step 2: List Multiples of the Second Integer

Write consecutive positive multiples of 18:
18, 36, 54, 72, 90...

3 Step 3: Identify Shared Values and Select the Lowest

Identify common multiples from both lists: {36, 72...}. The smallest positive matching number is 36. Therefore, $\text{LCM}(12, 18) = 36$.



Method 2: Prime Factorization & Highest Power Alignment

Prime factorization decomposes each number into its prime building blocks. The LCM is computed by taking every distinct prime factor raised to its greatest exponent found across the factorizations.

- Optimal for multi-digit numbers where listing multiples becomes tedious.
- Deconstruct each number to prime base powers: $12 = 2^2 \times 3^1$ and $18 = 2^1 \times 3^2$.
- Select the highest power for prime 2: $\max(2^2, 2^1) = 2^2 = 4$.
- Select the highest power for prime 3: $\max(3^1, 3^2) = 3^2 = 9$.
- Multiply the maximum prime powers: $2^2 \times 3^2 = 4 \times 9 = 36$.

1 Decompose 12 and 18

$12 = 2 \times 2 \times 3 = 2^2 \times 3^1$;
 $18 = 2 \times 3 \times 3 = 2^1 \times 3^2$.

2 Extract Peak Powers

Take 2^2 (highest power of 2) and 3^2 (highest power of 3).

3 Calculate Product

Evaluate the product: $2^2 \times 3^2 = 4 \times 9 = 36$.

Least Common Multiple (LCM)



Common Pitfall: Confusing LCM with GCF

Students frequently switch LCM (Least Common Multiple) and GCF (Greatest Common Factor). Because the word "Least" appears in LCM, learners mistakenly search for smaller values or select lowest exponents.

- GCF selects the lowest shared exponent: $2^1 \times 3^1 = 6$.
- LCM selects the highest present exponent: $2^2 \times 3^2 = 36$.
- Self-check verification rule: $\text{LCM}(a, b) \times \text{GCF}(a, b) = a \times b$.
For 12 and 18: $36 \times 6 = 216$, which equals $12 \times 18 = 216$.



Critical Difference Warning

LCM is never smaller than the largest input number ($\text{LCM} \geq \max(a, b)$).
GCF is never larger than the smallest input number ($\text{GCF} \leq \min(a, b)$).



Which prime exponent should you choose for the target operation?

Calculating LCM (Least Common Multiple)

Take each unique prime factor raised to its highest exponent across all factorizations.

Calculating GCF (Greatest Common Factor)

Take only common prime factors raised to their lowest exponent across all factorizations.

Fraction Foundations Mind Map



Core Definition: Part-to-Whole Anatomy

A fraction represents equal divisions of a defined whole unit or collection. The horizontal bar acts as a division operator separating two distinct integers.

$$\frac{3}{8}$$

Numerator (top integer):
Counts the number of equal fractional parts chosen, shaded, or considered.

Denominator (bottom integer):
Names the total count of equal parts that make up one full whole unit (where denominator is not zero).

Fraction Bar (Vinculum):

Represents mathematical division such that a/b is equivalent to a divided by b .



Visual Modeling Frameworks

Fractions can be represented across three primary visual frameworks to confirm equivalence and quantity.



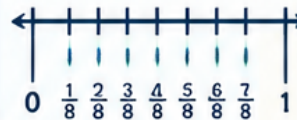
Area Model:

- A continuous two-dimensional region (circle, square, or rectangle) partitioned into congruent sub-regions.



Length Model (Fraction Strips):

- Linear bars of equal total length subdivided into congruent segments to compare magnitudes.



Number-Line Interval:

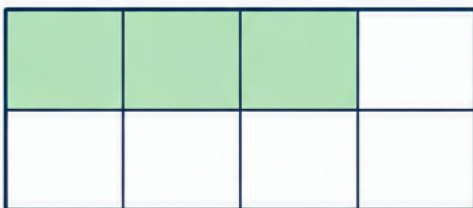
- Points along a continuous axis between whole integers, where the unit interval $[0, 1]$ is partitioned into equal sub-segments.

Fraction Foundations: Numerators & Denominators



Worked Example: Analyzing $3/8$ with an Area Grid

Follow these verified steps to construct and evaluate $3/8$ using a partitioned rectangular area model.



- Define the Whole Unit**
Draw a single unified rectangle representing 1 whole unit.
- Partition into Quadrants**
Divide the rectangle into 4 congruent columns, establishing 4 equal quadrants.
- Subdivide to Match Denominator**
Bisect each quadrant with a horizontal centerline, producing exactly 8 congruent sub-regions of identical area.
- Count and Shade the Numerator**
Shade exactly 3 of the 8 equal sub-regions to represent $3/8$ of the total area.



Error Analysis: Unequal Subdivisions

Fractional notation strictly requires that all subdivisions within the whole unit possess identical measure or area.

- Common Error:** Counting 3 shaded slices out of 8 non-congruent slices and writing $3/8$.
- Mathematical Reality:** Unequal parts cannot be counted directly by standard numerators. Subdivisions must be congruent in area or length before writing a standard fraction.



Common Mistake Warning

Counting parts only works when every part is equal in size. If a rectangle is cut into 2 huge pieces and 6 tiny pieces, shading 3 pieces does NOT equal $3/8$.



Partition Validation Decision Tree

Use this verification rule before applying standard fraction notation to any visual diagram.

Are all individual partitioned sectors within the whole model congruent in size and area?

Yes, all parts are strictly congruent
Count the shaded pieces for the numerator and the total pieces for the denominator to record the fraction a/b .

No, parts have different sizes or shapes
Do not count directly. Re-partition the shapes into equivalent smaller congruent subdivisions before assigning numerator and denominator values.



Topic Map: Equivalent Fractions & Simplification



Core Concept: The Multiplicative Identity Property

Equivalent fractions represent the exact same value or proportion of a whole, even though they use different numerators and denominators. This equivalence rests entirely on the Multiplicative Identity Property: multiplying or dividing any real number by 1 does not change its numerical value.

- Any non-zero quantity divided by itself equals 1 (for any $n \neq 0$, $n/n = 1$).
- Multiplying numerator and denominator by the identical factor n/n scales up without changing value: $\frac{a \cdot n}{b \cdot n} = \frac{a}{b}$.
- Dividing numerator and denominator by a shared common factor n/n simplifies without changing value: $\frac{a \div n}{b \div n} = \frac{a}{b}$.



The Golden Rule of Equivalence

Whatever operation (multiplication or division) you apply to the numerator, you must apply identically to the denominator. Adding or subtracting numbers does NOT preserve equivalence.



Simplification Method: Division by Greatest Common Factor (GCF)

A fraction is in its lowest terms (simplest form) when the numerator and denominator share no common factors other than 1. Dividing both terms by their Greatest Common Factor reduces the fraction in a single step.

- Identify all positive integer factors for both the numerator and the denominator.
- Select the largest factor present in both sets (the Greatest Common Factor).
- Divide both the numerator and denominator by this GCF to arrive at lowest terms.



Common Mistake: Asymmetrical Scaling

A frequent calculation error occurs when different operations or values are applied to the top and bottom terms. Operating unevenly alters the ratio and produces an unequal value.

- Error Example:** Dividing 18 by 6 (giving 3) but dividing 24 by 4 (giving 6), producing $\frac{3}{6} \neq \frac{12}{24}$.
- Additive Distortion:** Adding or subtracting the same number (e.g., $(18 - 6)/(24 - 6) = \frac{12}{18} = \frac{2}{3} \neq \frac{3}{4}$) does not maintain proportional equivalence.



Common Mistake Warning

Never use different divisors or multipliers! If numerator is divided by 6, the denominator must also be divided by 6. Using different numbers changes the value of the fraction.

Equivalent Fractions & Simplification

1

Step 1: Factor the Numerator and Denominator

For the fraction 18/24, list the integer factor pairs.
Factors of 18: 1, 2, 3, 6, 9, 18.
Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24.

2

Step 2: Determine the GCF

Compare common factors: {1, 2, 3, 6}.
The greatest common factor between 18 and 24 is 6.

3

Step 3: Divide Numerator and Denominator by GCF

Apply division by the identity factor (6/6):
 $18 \div 6 = 3$, and $24 \div 6 = 4$.

4

Step 4: Verify Lowest Terms

The resulting fraction is 3/4.
Factors of 3: {1, 3}; factors of 4: {1, 2, 4}.
Their only common factor is 1, confirming 3/4 is fully simplified.



Is your fraction completely reduced to lowest terms?

Numerator and denominator share a common factor greater than 1.

Divide both numbers again by that common factor or re-calculate the GCF.

Numerator and denominator have no common factor other than 1.

The fraction is in lowest terms; no further reduction is possible.



Self-Check Practice & Verification

Practice verifying equivalence and reducing to lowest terms using the worked rules.

- Test 1:** Find the GCF of 12 and 16, then simplify 12/16.
- Test 2:** Check whether 15/20 is equivalent to 3/4 by cross-multiplication ($15 \cdot 4 = 20 \cdot 3 = 60$).

Worked Reduction Practice:

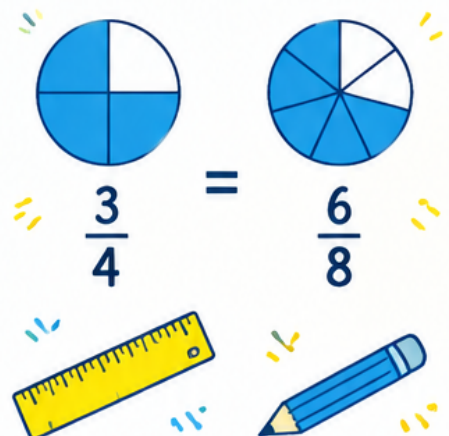
Simplify 20/30 (Record GCF and final lowest terms):

.....
.....

Equivalence Check:

Does $(9 \cdot 2) / (12 \cdot 2)$ maintain identity? Explain:

.....
.....



Unit 16: Proper, Improper Fractions and Mixed Number Conversions

1



Anatomy of Fractions and Mixed Numbers

A proper fraction has a numerator smaller than its denominator, representing a value strictly less than 1. An improper fraction has a numerator greater than or equal to its denominator, representing a value of 1 or greater. A mixed number expresses this quantity as a whole number combined with a proper fractional part.

- **Proper Fraction:** Numerator < Denominator (e.g., $\frac{3}{5}$)
- **Improper Fraction:** Numerator $\geq$ Denominator (e.g., $\frac{17}{5}$)
- **Mixed Number:** Whole number + Proper fraction (e.g., $3\frac{2}{5}$)

2



Converting Mixed Numbers to Improper Fractions

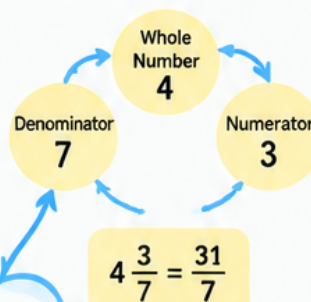
Follow the cyclic clockwise formula: multiply the denominator by the whole number, add the existing numerator to obtain the new numerator, and place the result over the unchanged denominator.

- Formula: (Denominator $\times$ Whole Number) + Numerator = New Numerator
- Retain the exact original denominator.

Step 1

Multiply Denominator by Whole Number

For the mixed number $4\frac{3}{7}$, identify the denominator 7 and whole number 4. Compute: $7 \times 4 = 28$.



Step 2

Add the Existing Numerator

Take the product 28 and add the fractional numerator 3: $28 + 3 = 31$.

Step 3

Reassemble Over the Denominator

Write the sum 31 over the original denominator 7, yielding the improper fraction $\frac{31}{7}$.

Fraction Conversions

4



Conversion Decision Path

Determine which mathematical conversion rule applies based on the given form of your fraction.

What is the form of the starting expression?

Starting with a Mixed Number ($A\frac{b}{c}$)

Apply cyclic formula: calculate $(c \times A) + b$ to form the new numerator, keeping denominator c to obtain improper fraction $\frac{(c \times A + b)}{c}$.

Starting with an Improper Fraction (N/D where $N \geq D$)

Execute integer division $N \div D$: the whole quotient Q forms the integer, and remainder R forms the proper fraction $\frac{R}{D}$, yielding $Q\frac{R}{D}$.

3



Converting Improper Fractions to Mixed Numbers

Use division to separate the improper fraction into whole units and a fractional remainder.

- Whole Number Part = Quotient from division (Dividend $\div$ Divisor)
- New Fractional Numerator = Remainder
- Denominator = Original Divisor (never changes)



Common Mistake: Inverted Fractional Remainder

Do not invert the fraction by writing the divisor over the remainder. When dividing 17 by 5, the remainder is 2 and the divisor is 5, giving $\frac{2}{5}$ ($3\frac{2}{5}$). Writing $3\frac{5}{2}$ is an improper remainder inversion.

Step 1 Divide Numerator by Denominator

For the improper fraction $\frac{17}{5}$, divide 17 by 5: $17 \div 5 = 3$ with a remainder of 2.

Step 2 Assign the Whole Number Quotient

The quotient 3 becomes the standalone whole number part of the mixed number.

Step 3 Form the Fractional Part

Place the remainder 2 over the original denominator 5, to form $\frac{2}{5}$, yielding the final mixed number $3\frac{2}{5}$.

$$\frac{17}{5} = 3\frac{2}{5}$$

Mind Map: Adding and Subtracting Fractions with Like Denominators



Core Rule: Fractions with Like Denominators

When fractions share the exact same denominator, they represent equal-sized fractional parts of a whole. Combine or take away the parts by calculating strictly across the top numbers while preserving the shared bottom number.

- Like denominators indicate that unit partition sizes match identically.
- Rule for addition: $\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$, where d does not equal 0.
- Rule for subtraction: $\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$, where d does not equal 0.



Denominator Invariance

The denominator names the total equal slices in one whole. Combining 5 twelfths and 4 twelfths yields 9 twelfths, not twenty-fourths.



Worked Numeric Example: Step-by-Step Procedure

Follow this three-stage sequence to solve and reduce fractions with identical denominators, demonstrated on $\frac{5}{12} + \frac{4}{12}$.

1 Verify Common Denominator

Confirm that both terms share the identical denominator value ($d = 12$).

2 Add Numerators

Add the top numbers together while retaining the common denominator:

$$\frac{5}{12} + \frac{4}{12} = \frac{5+4}{12} = \frac{9}{12}$$

3 Simplify to Lowest Terms

Find the greatest common factor of 9 and 12, which is 3. Divide both numerator and denominator by 3:

$$(\frac{9}{12} \div 3) = (\frac{9 \div 3}{12 \div 3}) = \frac{3}{4}$$



$\frac{5}{12}$



$\frac{4}{12}$



$\frac{9}{12}$

$$= \frac{3}{4}$$

Like Denominators: Add & Subtract



Common Pitfall Alert: Do Not Add Denominators

The most frequent error when adding like fractions is applying addition across both the top and bottom rews simultaneously.

- Incorrect calculation:** $\frac{5}{12} + \frac{4}{12} = \frac{5+4}{12+12} = \frac{9}{24}$.
- Why it fails:** $\frac{9}{24}$ reduces to $\frac{3}{8}$, which changes the scale of the whole instead of counting the twelfths.
- Correct calculation:** $\frac{5}{12} + \frac{4}{12} = \frac{9}{12} = \frac{3}{4}$.



Common Mistake Warning

NEVER calculate $12 + 12 = 24$ in the denominator. The denominator identifies what is being counted; it does not double when items are grouped.



Verification and Reduction Decision Check

Evaluate whether your combined fraction requires further factorization after performing the operation.



Checklist

- ☐ Final fraction check: shared factor between numerator and denominator identified ($\text{GCF} > 1$).
- ☐ Final simplified form written after division by GCF.

Do the numerator and denominator share any common factors greater than 1?

Yes, $\text{GCF} > 1$

Divide both numerator and denominator by their greatest common factor to reduce to simplest form.

No, $\text{GCF} = 1$

The fraction is already in simplest form and the calculation is complete.

Basic Math Mind Map:

Finding LCD and Renaming Fractions



Core Concept: Least Common Denominator (LCD)

To add or subtract fractions with unlike denominators, they must first share an identical base unit. The Least Common Denominator (LCD) is the Least Common Multiple (LCM) of the given denominators. Renaming preserves fractional value by scaling both numerator and denominator by the exact same non-zero factor.

- **Denominator:** The lower numeral indicating the total equal parts into which the whole is divided.
- **Least Common Multiple (LCM):** The smallest positive integer divisible by all given denominators.
- **Equivalence Rule:** Multiplying numerator and denominator by the same integer k (where $k > 0$) leaves the numeric fraction value unchanged.



Equivalence Principle

Multiplying a fraction by $\frac{k}{k}$ is multiplying by 1. It transforms the outward appearance to match denominators without altering the rational value.

Least Common Denominator (LCD) & Renaming



Step-by-Step LCD Workflow: Renaming $\frac{2}{3}$ and $\frac{3}{8}$

Follow this standard arithmetic sequence to rename two fractions with unlike denominators into equivalent fractions sharing their Least Common Denominator.

1 Step 1: Identify Original Denominators

Examine the fractions $\frac{2}{3}$ and $\frac{3}{8}$. The denominators are 3 and 8.

2 Step 2: Generate Multiples to Find the LCM

List positive multiples until the first common match appears:
 Multiples of 3: 3, 6, 9, 12, 15, 18, 21, 24, 27.
 Multiples of 8: 8, 16, 24, 32.
 The smallest common integer is 24, making LCD = 24.

3 Step 3: Determine Scaling Factors

Calculate the factor needed for each denominator to reach 24:
 For $\frac{2}{3}$, compute $\frac{24}{3} = 8$.
 For $\frac{3}{8}$, compute $\frac{24}{8} = 3$.

4 Step 4: Rename Each Fraction Equivalently

Multiply numerator and denominator by their respective scale factor:

$$\frac{2}{3} \times \frac{8}{8} = \frac{16}{24}$$

$$\frac{3}{8} \times \frac{3}{3} = \frac{9}{24}$$

5 Step 5: Verify Ready State for Operations

The pair is now $\frac{16}{24}$ and $\frac{9}{24}$. Because they share the identical common denominator 24, they can be directly added

$$\left(\frac{16}{24} + \frac{9}{24} = \frac{25}{24}\right) \text{ or subtracted } \left(\frac{16}{24} - \frac{9}{24} = \frac{7}{24}\right).$$



Common Mistake Alert: Straight Cross-Multiplication Without Denominator Adjustments

A frequent error occurs when students cross-multiply numerators directly across addends or simply add straight across numerators and denominators without establishing a valid shared denominator.

- **Incorrect Addition Across:** Writing $\frac{2}{3} + \frac{3}{8} = \frac{2+3}{3+8} = \frac{5}{11}$ is fundamentally invalid because thirds and eighths represent unequal partition sizes.
- **Incorrect Unadjusted Cross-Product:** Multiplying across terms (e.g., $2 \times 8 = 16$ and $3 \times 3 = 9$) without adjusting both fractions over the common base 24 leaves terms structurally mismatched.
- **The Correct Fix:** Always convert both terms to the common base first: $\frac{2}{3}$ becomes $\frac{16}{24}$, and $\frac{3}{8}$ becomes $\frac{9}{24}$. Then compute over 24.



Common Mistake Warning

Never add numerators together and denominators together directly. Fractions must represent equal partitions (common denominator) before numerators can be combined.



Mind Map: Adding and Subtracting Fractions with Unlike Denominators



Core Concept: The Common Denominator Rule

Fractions cannot be directly combined by addition or subtraction unless their denominators represent identical partition sizes. Operating across unlike denominators requires finding a common denominator to express each fraction with equivalent sharing units before executing the operation.

- **Denominator:** The bottom number specifying total equal parts in the whole.
- **Numerator:** The top number counting how many equal parts are taken.
- **Equivalence principle:** Multiplying both numerator and denominator by the same nonzero integer preserves the numerical value.

1 Stage 1: Find LCD

Identify the Least Common Multiple (LCM) of the given denominators. List multiples of each denominator or use prime factorization until the smallest matching value is determined.



Three-Stage Workflow: From Unlike to Like Denominators

Follow this standard sequence to add or subtract fractions with distinct denominators accurately.

1 Stage 1: Find the Least Common Denominator (LCD)

Identify the Least Common Multiple (LCM) of the given denominators. List multiples of each denominator or use prime factorization until the smallest matching value is determined.

2 Stage 2: Convert to Equivalent Fractions

Multiply both the numerator and the denominator of each fraction by the factor needed to turn each original denominator into the identified LCD.

3 Stage 3: Execute Arithmetic and Simplify

Add or subtract only the numerators while keeping the common denominator unchanged. Simplify the resulting fraction to lowest terms or convert an improper fraction if required.

Adding & Subtracting Fractions (Unlike Denominators)



Worked Solution: $\frac{3}{4}$ minus $\frac{2}{5}$

Problem: Evaluate $\frac{3}{4} - \frac{2}{5}$ step by step.

- **Step 1:** Determine LCD for 4 and 5. Multiples of 4 are 4, 8, 12, 16, 20... Multiples of 5 are 5, 10, 15, 20... The LCD is 20.
- **Step 2:** Scale fractions to common denominator 20: $(3 \times 5) / (4 \times 5) = 15/20$, and $(2 \times 4) / (5 \times 4) = 8/20$.
- **Step 3:** Subtract numerators across the common denominator: $15/20 - 8/20 = (15 - 8) / 20 = 7/20$.

Result: $7/20$ (already in simplest form because 7 is prime and not a factor of 20).



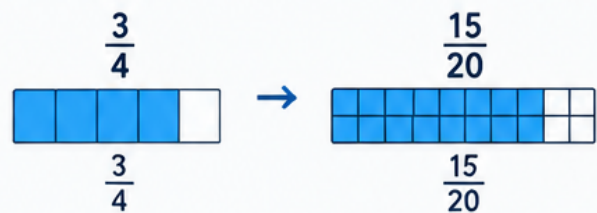
Common Mistake Warning

Never subtract or add numerators and denominators straight across (for example, computing $(3 - 2)/(4 - 5) = 1/(-1)$ is completely incorrect). Denominators establish partition scale and must remain unified.



Stage 2: Convert Equivalent Fractions

Multiply both the numerator and the denominator of each fraction by the factor needed to turn each original denominator into the identified LCD.



Self-Check Inspection Rules

Apply this quick check whenever completing fraction addition or subtraction.

- ☐ Are both denominators identical before adding or subtracting?
- ☐ Did you multiply both top and bottom by the exact same scale factor?
- ☐ Did the denominator stay constant in your final calculated step?
- ☐ Is the final fraction reduced to lowest terms?





Adding and Subtracting Mixed Numbers with Regrouping Mind Map



Core Concept: Regrouping Mixed Numbers

When subtracting mixed numbers where the top fraction is smaller than the bottom fraction, borrow 1 from the whole number. Convert that whole number into an equivalent fraction with the same denominator (d/d) and add it to the existing numerator.

- Identify the common denominator before comparing fractional parts.
- Borrow 1 from the whole number minuend: W becomes $(W - 1)$.
- Express 1 as d/d matching the shared denominator.
- Combine the borrowed unit with the existing fraction: $n/d + d/d = (n + d)/d$.
- Subtract whole parts and fractional parts separately, then simplify.



Denominator Base Rule

Unlike whole-number borrowing which operates in base 10, fractional regrouping operates in base d . Always add the value of the denominator to the numerator, not 10.



Step-by-Step Regrouping Procedure

Follow this standard sequence to subtract mixed numbers requiring fractional borrowing.

- Check Common Denominators**
Ensure both fractions share identical denominators. If different, determine the LCM and find equivalent fractions.
- Compare Numerators**
Determine if the minuend fraction is smaller than the subtrahend fraction. If regrouping is required, proceed to borrow.
- Regroup 1 Whole into Fractional Form**
Decrease the whole number by 1. Add d/d to the fractional part, forming an improper fraction: $W \frac{n}{d}$ becomes $(W - 1) \frac{n + d}{d}$.
- Subtract Fractions and Wholes**
Subtract the fractional numerators: $(n1 - n2)/d$. Then subtract the remaining whole numbers.
- Reduce to Simplest Terms**
Divide the resulting numerator and denominator by their greatest common factor (GCF) if applicable.



Worked Exemplar:

$$5 \frac{1}{4} - 2 \frac{3}{4}$$

Follow the complete computation demonstrating fractional regrouping and simplification.

- Initial problem:** $5 \frac{1}{4} - 2 \frac{3}{4}$
- Compare numerators:** $\frac{1}{4} < \frac{3}{4}$, regrouping is required.
- Decompose whole number:** $5 = 4 + 1 = 4 + \frac{4}{4}$.
- Add to existing fraction:** $4\frac{4}{4} + \frac{1}{4} = 4\frac{5}{4}$, yielding $4 \frac{5}{4}$.
- Perform subtraction:** $(4 - 2) + (\frac{5}{4} - \frac{3}{4}) = 2 \frac{2}{4}$.
- Reduce fractional component:** $\frac{2}{4} = \frac{1}{2}$ (dividing numerator and denominator by 2).
- Final Result:** $2 \frac{1}{2}$.

$$5 \frac{1}{4} - 2 \frac{3}{4} = 4 \frac{5}{4} - 2 \frac{3}{4} = 2 \frac{2}{4} = 2 \frac{1}{2}$$



Common Mistake Analysis

Watch for standard arithmetic traps during regrouping.

- Base-10 Borrowing Error:** Writing $5 \frac{1}{4}$ as $4 \frac{11}{4}$ by adding 10 to the numerator instead of adding the denominator 4.
- Reversing Subtraction Order:** Subtracting $\frac{1}{4}$ from $\frac{3}{4}$ to get $2 \frac{3}{4}$ without borrowing from the whole number.
- Forgotten Whole Number Reduction:** Adding $\frac{4}{4}$ to the fraction while keeping the whole number 5 unchanged (producing $5 \frac{5}{4}$).
- Skipping Final Simplification:** Leaving $2 \frac{2}{4}$ unreduced instead of dividing by the GCF to reach $2 \frac{1}{2}$.



Critical Error Alert

Borrowing 1 whole always adds the denominator value d , never 10. For fourths, $+1 \text{ whole} = +4/4$.

Regrouping Mixed Numbers



Visual Decision Point

Is the top numerator smaller than the bottom numerator?

Yes

(Fractional subtraction cannot proceed directly)

Borrow 1 from the whole number, convert it to d/d , add it to the top numerator to make an improper fraction, then subtract.

No

(Top numerator is greater than or equal to bottom numerator)

Subtract the numerators directly and subtract the whole numbers without regrouping.



Guided Student Practice

Complete the practice computations below using the regrouping method.

1. Solve $6 \frac{1}{5} - 2 \frac{4}{5}$: Show regrouping of $6 \frac{1}{5}$ and final simplified answer.

2. Solve $4 \frac{2}{7} - 1 \frac{5}{7}$: Show improper fraction regrouping and calculation steps.

3. Error Check: Explain why rewriting $7 \frac{1}{6}$ as $6 \frac{11}{6}$ is incorrect.

Multiplying Proper and Improper Fractions Mind Map



Multiplying Proper and Improper Fractions

Fraction multiplication combines parts of parts across numerators and denominators without requiring a common denominator. The general rule multiplies top numbers together and bottom numbers together:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

- Multiply numerators straight across to form the new numerator.
- Multiply denominators straight across to form the new denominator.
- Common denominators are never required when multiplying fractions.
- Applicable directly to both proper fractions and improper fractions.



Core Formula

For real numbers a, b, c, d with $b \neq 0$ and $d \neq 0$:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$



Visual Area Model Representation

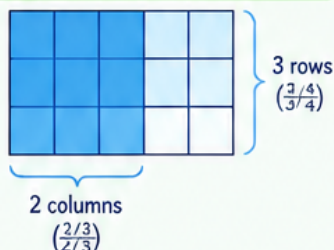
Multiplying fractions corresponds to finding the overlapping area of a unit square partitioned by horizontal and vertical division lines.

- Partition a unit square vertically into columns equal to the first denominator.
- Partition the square horizontally into rows equal to the second denominator.
- Shade columns corresponding to the first numerator and rows for the second numerator.
- The doubly shaded intersection represents the resulting product out of the total grid cells.



Visual Check: $\frac{2}{3} \times \frac{3}{4}$

A 3-by-4 grid produces 12 equal parts. Shading 2 vertical columns and 3 horizontal rows yields 6 overlapping cells:
 $\frac{6}{12} = \frac{1}{2}$



Cross-Simplification Strategy

Cross-simplification reduces numbers before carrying out multiplication, keeping values smaller and preventing tedious factoring after the operation.

- Pair any numerator with any denominator to find common factors.
- Divide both values by their Greatest Common Factor (GCF) before multiplying.
- Repeat across both diagonal pairings until no common factor remains besides 1.

1

Identify Diagonal or Vertical Common Factors

Check if a numerator and an opposing denominator share a factor greater than 1.

2

Divide by the Greatest Common Factor

Cancel and replace the original values with their quotients.

3

Multiply Remaining Values Straight Across

Multiply the simplified numerators and simplified denominators.

Multiplying Fractions



Worked Numeric Examples

Step-by-step solutions demonstrating proper and improper fraction products with cross-simplification and conversion.

Example A (Proper Fractions):

$\frac{3}{8} \times \frac{4}{9}$ → simplify 3 and 9 by 3
 (leaves 1 and 3): simplify 4 and 8 by 4
 (leaves 1 and 2) → $\frac{1 \times 1}{2 \times 3} = \frac{1}{6}$

Example B (Improper Fractions):

$\frac{7}{4} \times \frac{8}{5}$ → simplify 8 and 4 by 4
 (leaves 2 and 1) → $\frac{7 \times 2}{1 \times 5} = \frac{14}{5} = 2\frac{4}{5}$



Improper to Mixed Form

When the product numerator is larger than the denominator, divide the numerator by the denominator: $\frac{14}{5} = 2$ with a remainder of 4, yielding $2\frac{4}{5}$.



Common Mistakes and Verification Decision

Avoid frequent fraction arithmetic traps by following standard reduction protocols.

- **Mistake:** Finding a common denominator before multiplying. This creates unnecessarily large values and wastes time.
- **Mistake:** Cross-multiplying as if solving a proportion equation. Cross-multiplying is only for $\frac{a}{b} = \frac{c}{d}$, not $\frac{a}{b} \times \frac{c}{d}$.
- **Mistake:** Canceling between two numerators or between two denominators instead of between a top and a bottom value.



Common Mistake Alert

Never invert the second fraction during multiplication! Inverting is reserved exclusively for fraction division.



Is your resulting fraction in simplest form or an improper fraction?

Yes

Numerator and denominator share a common factor $\times 1$

Divide both top and bottom by their greatest common factor to reduce to lowest terms.

No

Numerator is greater than denominator and a mixed number is requested

Divide numerator by denominator; write quotient as whole number and remainder over the original denominator.



Dividing Fractions & Reciprocal Multiplication Mind Map



1 Visual Concept & Grouping

3 pizzas divided into halves:



3 wholes = 6 halves

4 Worked Numeric Examples

Step-by-step solutions demonstrating the division of unit fractions, proper fractions, and whole numbers.

Example 1 (Unit Fraction Divisor):

$$\frac{3}{5} \div \frac{1}{4} = \frac{3}{5} \times \frac{4}{1} = \frac{12}{5} = 2 \frac{2}{5}$$

Example 2 (Non-Unit Fractions):

$$\frac{2}{3} \div \frac{4}{9} = \frac{2}{3} \times \frac{9}{4} = \frac{18}{12} = \frac{3}{2} = 1 \frac{1}{2}$$

Example 3 (Whole Number by Fraction):

$$6 \div \frac{2}{3} = \frac{6}{1} \times \frac{3}{2} = \frac{18}{2} = 9$$

Example 4 (Fraction by Whole Number):

$$\frac{3}{8} \div 2 = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$$

2 Understanding the Reciprocal (Multiplicative Inverse)

The reciprocal of any non-zero number is the number that, when multiplied by the original value, yields a product of 1. For any fraction $\frac{a}{b}$ (where a and b are non-zero), its reciprocal is $\frac{b}{a}$.

- Proper fraction reciprocal: $\frac{3}{4} \rightarrow \frac{4}{3}$
because $(\frac{3}{4}) \times (\frac{4}{3}) = \frac{12}{12} = 1$.
- Whole number reciprocal: $5 \rightarrow \frac{1}{5}$.
- Unit fraction reciprocal: $\frac{1}{6} \rightarrow \frac{6}{1} = 6$.
- Mixed number reciprocal: Convert mixed numbers into improper fractions first.

3 The Keep-Change-Flip Algorithm

The standard procedure for dividing any two fractions converts the division problem into a direct fraction multiplication using the reciprocal of the divisor.

- 1 Keep the Dividend**
Leave the first fraction exactly as it is written: Keep $\frac{a}{b}$.
- 2 Change the Operation**
Change the division sign ($\div$) to a multiplication sign ($\times$).
- 3 Flip the Divisor**
Replace the second fraction (the divisor) with its reciprocal: Flip $\frac{c}{d}$ to $\frac{d}{c}$.
- 4 Multiply and Simplify**
Multiply numerators straight across ($a \times d$), multiply denominators straight across ($b \times c$), and reduce to lowest terms.

DIVIDING FRACTIONS

5 Common Mistakes & How to Avoid Them

Fraction division errors frequently stem from flipping the wrong term or confusing operations.

- **Never flip the first fraction:** Keep the dividend unchanged. Only flip the divisor (the fraction following the $\div$ sign).
- **Never invert both fractions:** Flipping both values changes the mathematical relationship and produces an incorrect result.
- **Do not cross-multiply before rewriting:** Convert division to multiplication first to ensure correct numerator and denominator alignment.
- **Convert mixed numbers first:** Invert only after rewriting mixed numbers as improper fractions (e.g., convert $1\frac{12}{12}$ to $\frac{32}{12}$, then invert to $\frac{12}{32}$).

⚠ Critical Warning

Flipping the dividend instead of the divisor produces the reciprocal of the correct quotient. Always verify that only the second fraction was inverted.

Which fraction should you invert when computing a division problem?

The first fraction (Dividend)

Do not invert. Keep the dividend intact.

The second fraction (Divisor)

Correct. Invert the divisor to its reciprocal and multiply.

6 Student & Tutor Practice Check

Work through these verification problems using the Keep-Change-Flip method.

1 Problem 1: $\frac{5}{6} \div \frac{2}{3} =$

Keep: _____ $\times$ Flip: _____ = Answer: _____

2 Problem 2: $4 \div \frac{1}{3} =$

Keep: _____ $\times$ Flip: _____ = Answer: _____

3 Problem 3: $\frac{7}{10} \div 5 =$

Keep: _____ $\times$ Flip: _____ = Answer: _____



Multiplying and Dividing Mixed Numbers Mind Map



Step 1: Convert Mixed Numbers to Improper Fractions

Before performing multiplication or division, every mixed number must be converted into an improper fraction. Multiply the whole number by the denominator, add the numerator to get the new numerator, and keep the original denominator unchanged.

Formula: $A \frac{b}{c} = \frac{A \times c + b}{c}$

- **Example:** $2 \frac{1}{4} = \frac{(2 \times 4 + 1)}{4} = \frac{9}{4}$
- **Example:** $1 \frac{2}{3} = \frac{(1 \times 3 + 2)}{3} = \frac{5}{3}$
- Never multiply or divide whole parts and fraction parts separately.



Core Rule

Direct arithmetic on mixed numbers without conversion leads to severe errors. Always rewrite mixed numbers as improper fractions first.

Steps:

- 1 **Multiply Whole by Denominator**
For $2 \frac{3}{5}$, multiply the whole number 2 by the denominator 5 to obtain 10.
- 2 **Add the Numerator**
Add the numerator 3 to 10 to obtain the improper numerator 13.
- 3 **Retain the Denominator**
Place 13 over the original denominator 5 to form $\frac{13}{5}$.



Step 2: Operation Choice — Multiply vs. Divide

Once both terms are in improper fractional form, determine the appropriate operational pathway for computation.

- **Multiplication:** Directly multiply numerators together and denominators together.
- **Cross-simplification:** Divide out common factors diagonally or vertically before multiplying to work with smaller numbers.
- **Division:** Invert the second fraction (the divisor) to find its reciprocal, then change the division sign to multiplication.

Which operation is being performed between the two improper fractions?

Multiplication ($a/b \times c/d$)

Check for common factors to cross-simplify diagonally, then multiply numerators straight across ($a \times c$) and denominators straight across ($b \times d$).

Division ($a/b \div c/d$)

Keep the first fraction, change the division sign to multiplication, invert the second fraction to its reciprocal (d/c), cross-simplify if possible, and multiply.

Multiplying & Dividing Mixed Numbers



Worked Numeric Example: Multiplication

Evaluate the product: $2 \frac{1}{4} \times 1 \frac{1}{3}$.

- 1 **Step 1:** Convert $2 \frac{1}{4} = \frac{(2 \times 4 + 1)}{4} = \frac{9}{4}$.
- 2 **Step 2:** Convert $1 \frac{1}{3} = \frac{(1 \times 3 + 1)}{3} = \frac{4}{3}$.
- 3 **Step 3:** Set up the product: $\left(\frac{9}{4}\right) \times \left(\frac{4}{3}\right)$.
- 4 **Step 4:** Cross-cancel common factors: divide 4 and 4 by 4 (both become 1); divide 9 and 3 by 3 (9 becomes 3, 3 becomes 1).
- 5 **Step 5:** Multiply remaining terms:
 $(3 \times 1) \times (1 \times 1) = \frac{3}{1} = 3$.



Cross-Simplification Tip

Cross-cancelling before multiplying avoids large products and simplifies reducing the final result.



Step 3: Convert Back to Simplified Mixed Number

If the final product or quotient is an improper fraction with a numerator greater than its denominator, convert it back into a mixed number in simplest terms.

- Divide the numerator by the denominator.
- The whole-number quotient becomes the whole part of the mixed number.
- The remainder becomes the new numerator over the original denominator.
- Verify that the fractional part cannot be reduced further.



Common Mistake

Do not invert the first fraction (dividend) during division. Only invert the second fraction (divisor).

Steps:

- 1 **Divide Numerator by Denominator**
For $\frac{11}{4}$, calculate $11 \div 4 = 2$ with a remainder of 3.
- 2 **Construct Mixed Number**
Combine the quotient and remainder: 2 with remainder 3 over 4 gives $2 \frac{3}{4}$.
- 3 **Final Reduction Check**
Ensure greatest common factor of remainder and denominator is 1. Since $\text{GCF}(3, 4) = 1$, $2 \frac{3}{4}$ is fully simplified.



Worked Numeric Example: Division

Evaluate the quotient: $3 \frac{1}{2} \div 1 \frac{3}{4}$.

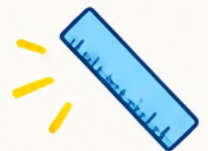
- 1 **Step 1:** Convert $3 \frac{1}{2} = \frac{(3 \times 2 + 1)}{2} = \frac{7}{2}$.
- 2 **Step 2:** Convert $1 \frac{3}{4} = \frac{(1 \times 4 + 3)}{4} = \frac{7}{4}$.
- 3 **Step 3:** Rewrite division as multiplication by reciprocal: $\left(\frac{7}{2}\right) \div \left(\frac{7}{4}\right) = \left(\frac{7}{2}\right) \times \left(\frac{4}{7}\right)$.
- 4 **Step 4:** Cross-cancel: divide 7 and 7 by 7 (both become 1); divide 4 and 2 by 2 (4 becomes 2, 2 becomes 1).
- 5 **Step 5:** Multiply remaining numerators and denominators: $(1 \times \frac{2}{1}) \times (1 \times 1) = \frac{2}{1} = 2$.

$$\frac{7}{2} \times \frac{4}{7} \rightarrow \frac{1}{1} \times \frac{2}{1} = 2$$





Mind Map: Decimal Place Value and Base-Ten Expansion



Core Concept: The Decimal Place-Value System

The decimal point separates the whole number part from the fractional part. In our base-ten system, moving one position to the right divides the place value by 10, meaning each decimal position represents a fraction whose denominator is an increasing power of ten.

- Tenths ($1/10$ or 0.1): First position immediately right of the decimal point.
- Hundredths ($1/100$ or 0.01): Second position right of the decimal point.
- Thousandths ($1/1000$ or 0.001): Third position right of the decimal point.
- Ten-thousandths ($1/10000$ or 0.0001): Fourth position right of the decimal point.



Base-Ten Shift Rule

Each step right decreases place value by a factor of 10 (divided by 10). Each step left increases place value by a factor of 10 (multiplied by 10).



Positional Chart and Base-Ten Expansion Models

Every decimal number can be decomposed into expanded form using fraction equivalents or decimal values.

- Example Value: 42.375
- Expanded Form (Fractions):
 $40 + 2 + 3/10 + 7/100 + 5/1000$
- Expanded Form (Decimals):
 $40 + 2 + 0.3 + 0.07 + 0.005$
- Expanded Form (Multiplicative):
 $(4 \times 10) + (2 \times 1) + (3 \times 0.1) + (7 \times 0.01) + (5 \times 0.001)$

Positional Chart

Tens (10)	Ones (1)	Tenths (0.1)	Hundredths (0.01)	Thousandths (0.001)
4	2	3	7	5

- Step 1: Identify Whole vs Fractional Portions**
Locate the decimal point. Values to the left are whole units (tens = 4, ones = 2). Values to the right are fractional parts (tenths = 3, hundredths = 7, thousandths = 5).
- Step 2: Assign Place-Value Multipliers**
Multiply each digit by its respective place value:
 $4 \times 10, 2 \times 1, 3 \times 1/10, 7 \times 1/100, 5 \times 1/1000$.
- Step 3: Combine in Summation**
Write the complete expanded expression:
 $40 + 2 + 0.3 + 0.07 + 0.005 = 42.375$.



Comparing and Ordering Decimals

To compare decimal magnitudes, align the decimal points and evaluate digits sequentially from left to right across corresponding place values.

- Compare 0.42 and 0.375: Both have 0 ones. Look at the tenths place: 4 tenths is greater than 3 tenths ($4 > 3$). Therefore, $0.42 > 0.375$.
- Compare 0.50 and 0.5: Both possess 0 ones and 5 tenths. Adding trailing zeros to the right does not alter decimal magnitude, so $0.50 = 0.5$.



Common Mistake: Longer Does Not Mean Greater

Do not treat decimals as whole numbers. For example, 0.375 is not greater than 0.42 simply because 375 is larger than 42; tenths take precedence over thousandths.



Decision: Which decimal is larger: 0.608 or 0.62?

- 1 Compare ones and tenths: Ones match ($0 = 0$) and tenths match ($6 = 6$). Move to hundredths. Examine the hundredths position: 0.608 has 0 hundredths, while 0.62 has 2 hundredths.
- 2 Since 2 hundredths is strictly greater than 0 hundredths ($2 > 0$): Conclude that $0.62 > 0.608$, regardless of the additional digit in the thousandths position.

DECIMAL PLACE VALUE



Practice and Diagnostic Checks

Apply positional analysis and expanded notation to standard decimal values.

- Decompose 5.084 into base-ten expanded fractional form.
- Identify the digit in the thousandths place of 19.3248.
- Insert $>$, $<$, or $=$ between 0.079 and 0.08.

5.084 Expanded Form

Thousandths Digit of 19.3248

Comparison: 0.079 [?] 0.08

Small steps
build big skills!



Adding and Subtracting Decimals with Decimal Point Alignment



Core Concept: Decimal Point Alignment

When adding or subtracting decimal numbers, vertical alignment is required to ensure that digits with matching place values are combined. The decimal point acts as the structural anchor. Aligning the points guarantees tenths add to tenths, hundredths add to hundredths, and whole number place values remain properly positioned.

- Align all decimal points vertically in a straight column before computing.
- Maintain identical column positioning down into the sum or difference line.
- Add trailing zero placeholders to fill empty positions to the right of the decimal point.
- Perform regrouping (carrying or borrowing) across the decimal point following standard base-ten rules.



Core Alignment Rule

Never align decimal numbers by their rightmost digit. Always line up the decimal points directly above one another.



Step-by-Step Procedure

Follow this four-step sequence to set up, balance, and calculate any decimal addition or subtraction problem accurately.

- 1 Stack Vertically by Decimal Point**
Write the first addend or minuend. Place the second number directly underneath so both decimal points form a straight vertical line.
- 2 Pad with Trailing Zero Placeholders**
If one number has fewer decimal places than the other, annex trailing zeros to the right so both numbers have the same number of decimal digits.
- 3 Bring Down the Decimal Point**
Place a decimal point directly below the column of points into the answer space before calculating.
- 4 Add or Subtract from Right to Left**
Compute each column from right to left, regrouping across whole number and decimal boundaries as needed.

Adding & Subtracting Decimals



Worked Numeric Examples

Carefully observe how trailing zero placeholders enable regrouping across the decimal boundary in subtraction and maintain column integrity in addition.

- **Addition Setup:** $14.8 + 3.26$ becomes $14.80 + 03.26 = 18.06$.
- **Subtraction Setup:** $23 - 0.37$ becomes $23.00 - 00.37 = 22.63$.
- **Regrouping Note:** In $23.00 - 0.37$, regroup 1 from the ones column (reducing 3 to 2), convert to 10 tenths, then regroup 1 tenth into 10 hundredths (leaving 9 tenths and 10 hundredths: $10 - 7 = 3$, $9 - 3 = 6$, $2 - 0 = 2$, $2 - 0 = 2$).

Addition: $14.80 + 03.26 = 18.06$

Hundreds	Tens	Ones	Tenths	Hundredths
1	4	.	8	0
+	0	3	.	2
1	8	.	0	6

Subtraction: $23.00 - 00.37 = 22.63$

Hundreds	Tens	Ones	Tenths	Hundredths
2	3	.	0	0
-	0	0	.	3
2	2	.	6	3



Whole Number Decimal Representation

Any whole integer has an implied decimal point immediately to its right. Write 23 as 23.00 when subtracting hundredths.



Common Mistakes and Misconceptions

Review these critical errors before completing decimal calculations to avoid standard pitfalls.

- **Ragged-Right Alignment:** Lining numbers up flush against the right edge (like whole-number multiplication) instead of matching the decimal point.
- **Subtracting from Blank Space:** Writing a bottom digit into the answer without annexing a top zero and regrouping (e.g., computing $5.4 - 2.18$ and incorrectly writing 8 in the hundredths place instead of $5.40 - 2.18 = 3.22$).
- **Floating Points:** Placing the decimal point arbitrarily in the answer instead of dropping it straight down into the corresponding column.



Common Mistake Alert

Do not treat blank positions in the minuend as self-fulfilling digits. In subtraction, a missing place value must be represented by 0 before regrouping.



Diagnostic Check: Alignment vs. Ragged Right

Use this quick decision guide to verify your setup before computing sums or differences.

Are all decimal points lined up in a single vertical column?



Yes, decimal points form a straight vertical line.

Check if each row has equal decimal places. If uneven, annex trailing zeros to the right, drop the decimal point directly into the answer space, and compute from right to left.



No, digits are aligned flush to the right margin

Erase and rewrite the problem immediately. Align by the decimal points regardless of whether the right or left margins appear jagged.



Multiplying Decimals Mind Map Card



Core Principle: Integer Multiplication and Decimal Summation

When multiplying numbers with decimals, ignore the decimal points initially and multiply the factors as if they were standard whole integers. The total count of decimal places in the product is equal to the sum of the decimal places in each factor.

- Ignore decimal points during vertical column multiplication.
- Count the digits to the right of the decimal point in the first factor.
- Count the digits to the right of the decimal point in the second factor.
- Add both counts together to determine total decimal shifts needed in the answer.



Decimal Alignment Rule

Unlike addition and subtraction, decimal points do NOT need to be aligned in columns when setting up a multiplication problem.

Steps:

- Step 1: Multiply as Integers**
Set up the multiplication without worrying about decimal alignment. Multiply the numbers digit by digit exactly as whole numbers.
- Step 2: Total the Decimal Places**
Count the number of decimal digits after the point in both factors and sum them.
- Step 3: Position the Decimal Point**
Starting from the rightmost digit of your product, count left the exact total number of places calculated in Step 2, then place the decimal point.



Worked Numeric Example: Standard Factors

Detailed step-by-step calculation showing how to multiply 3.72 by 4.5.

- Factor 1: 3.72 has 2 decimal places.
- Factor 2: 4.5 has 1 decimal place.
- Total decimal places needed in product: $2 + 1 = 3$ decimal places.

1 Integer Multiplication

Multiply 372 by 45:

$$372 \times 5 = 1,860$$

$$372 \times 40 = 14,880$$

$$1,860 + 14,880 = 16,740.$$

2 Decimal Shift

Place the decimal point 3 places from the right: 16,740 becomes 16.740, which simplifies to 16.74.

$$\begin{array}{ccc} 3.72 & \times & 4.5 & = & 16.740 \\ \underbrace{}_{2 \text{ places}} & & \underbrace{}_{1 \text{ place}} & & \underbrace{}_{3 \text{ places}} \end{array}$$



Special Case: Inserting Leading Zeros

When the integer product contains fewer digits than the required number of decimal places, insert leading zeros directly to the left of the product before placing the decimal point.

- Example calculation: 0.03×0.2
- Whole number multiplication: $3 \times 2 = 6$.
- Factor 0.03 has 2 decimal places; factor 0.2 has 1 decimal place. Total needed: 3 places.
- Prefix two zeros before the digit 6 to create three decimal positions: 0.006.



Common Mistake Alert

Do not append zeros to the right of the product when you run out of digits. Appending zeros at the end changes the value incorrectly (0.600 instead of 0.006). Always prefix zeros to the left.



Does the integer product have fewer digits than the total decimal places counted?

Yes (Fewer digits)

Prefix placeholder zeros immediately to the left of the integer product until the count matches, then add the decimal point and a leading zero.

No (Equal or more digits)

Count from the far right digit toward the left by the total places and place the decimal point directly between the appropriate digits.



Self-Check and Practice Verification

Follow this checklist to verify your solution every time you multiply decimal numbers.

- ☒ Did I multiply without forcing decimal point alignment?
- ☒ Did I count the total decimal places from both factors accurately?
- ☒ Did I shift left from the rightmost digit rather than starting from the left?
- ☒ If digits were insufficient, did I insert leading zeros on the left?
- ☒ Is the magnitude of the final answer reasonable based on mental estimation?



Practice

Problem Expression to Solve

Factor 1 Decimal Count + Factor 2 Decimal Count

Calculated Integer Product

Final Product with Placed Decimal



Dividing Decimals Mind Map Card



Dividing Decimals Overview

Division with decimal values relies on place value shifts to transform divisors into whole numbers, ensuring predictable standard algorithm steps.

- A decimal dividend represents parts of a whole separated by a point.
- Dividing by a whole number preserves the decimal position directly in the quotient.
- Dividing by a decimal requires scaling both numbers by matching powers of ten.



Decimal Point Alignment Rule

Always place the quotient decimal point directly above the final adjusted dividend point before computing digits.



Dividing by a Whole Number

When the divisor is already a whole number, division proceeds identically to standard integer long division, raising the decimal directly upward.

- Align the quotient decimal directly above the dividend decimal.
- Divide sequentially from left to right.
- Append trailing zeros to the dividend if a remainder persists.

Step 1:

Set Up and Align

Write the problem in long division format. Place the decimal point on the quotient bar directly above the dividend point.

Step 2:

Divide Normal Integers

Compute quotient digits as with whole numbers, ignoring the point during place-to-place subtraction.

Step 3:

Annex Zeros if Needed

If a remainder remains after using all dividend digits, append zeros to the right of the decimal dividend and continue.

Dividing Decimals



Dividing by a Decimal Divisor

To divide by a decimal divisor, shift the decimal point rightward in both divisor and dividend by identical places until the divisor is an integer.

- Count the decimal places in the divisor.
- Multiply both divisor and dividend by 10, 100, or 1,000 depending on that count.
- Execute standard long division with the transformed whole-number divisor.



Equal Shift Verification

Never move the dividend decimal fewer or more places than the divisor decimal; multiplying by identical powers of 10 keeps the fraction equivalent.

Does the divisor contain a decimal point?

Divisor is already a whole number

Place the quotient decimal point directly above the dividend decimal point and divide normally.

Divisor has one or more decimal places

Shift the decimal point to the right until the divisor is a whole number, shift the dividend point by the identical number of places, and divide.



Worked Numeric Example & Common Pitfalls

Detailed demonstration of 0.42 divided by 1.2 , showing power-of-ten equivalence and digit placement.

- Expression: $0.42 \div 1.2$
- Shift 1 place right: $4.2 \div 12$
- Quotient decimal placed above 4.2
- Calculation: 12 goes into 4 zero times, into 42 three times (36, remainder 6), into 60 five times (60, remainder 0)
- Final Result: 0.35

$$\begin{array}{r} 0.35 \\ 12 \overline{) 42.0} \\ \underline{- 0} \\ 42 \\ \underline{- 36} \\ 6.0 \\ \underline{- 60} \\ 0 \end{array}$$



Common Mistake

Forgetting to add a trailing zero in the dividend to finish dividing remainders, or misaligning quotient place values after shifting.

Practice Problem: Evaluate $1.56 \div 0.6$

Checked Answer & Verification Notes

Converting Between Fractions, Decimals, and Percentages



Conversion Pathways Overview

Fractions, decimals, and percentages represent the exact same numeric value in three distinct mathematical notations. Converting between them relies on powers of ten and standard division algorithms.

- Fractions express a part-to-whole ratio using an integer numerator divided by a non-zero denominator.
- Decimals use base-ten positional place values (tenths, hundredths, thousandths) separated by a decimal point.
- Percentages represent parts per hundred, designated by the percent symbol (%).



Equivalence Principle

Every finite or repeating decimal corresponds to a rational fraction, and every percentage is equivalent to a fraction with a denominator of 100.



Converting Fractions to Decimals and Percentages

To change any fraction into an equivalent decimal or percentage, perform division or scale to a power of ten.

- Fractions to Decimals:** Divide the numerator by the denominator using long division until the remainder terminates or repeats.
- Fractions to Percentages:** First convert to a decimal and multiply by 100%, or multiply the fraction directly by 100%.

Step 1: Divide Numerator by Denominator

For the fraction $\frac{3}{8}$, compute 3 divided by 8. Long division yields 0.375.

Step 2: Scale to Percentage

Multiply 0.375 by 100 and affix the percent symbol: $0.375 \times 100 = 37.5\%$.



Converting Percentages to Decimals and Fractions

Because percent literally denotes 'per one hundred', conversion begins by removing the symbol and dividing by 100.

- Percentages to Decimals:** Divide by 100 (shift the decimal point two places to the left) and drop the % sign.
- Percentages to Fractions:** Place the percentage value over 100, remove the % sign, and reduce to simplest form.

Step 1: Percentage to Decimal

For 58%, compute $58 \div 100 = 0.58$.

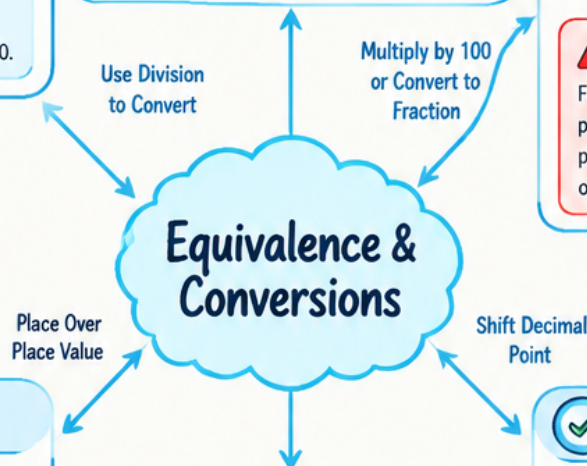
Step 2: Percentage to Fraction

Write $58/100$ and simplify by dividing by GCF 2 to obtain $29/50$.



Common Mistake Alert

Forgetting to shift the decimal point two full places when converting single-digit or decimal percentages (e.g., writing 5% as 0.5 instead of 0.05, or 0.4% as 0.4 instead of 0.004).



Decimals

Decimals can be converted directly into fractions based on positional place value, and into percentages by shifting the decimal point.

- Decimals to Fractions:** Read the decimal place value (e.g., tenths = $\frac{1}{10}$, hundredths = $\frac{1}{100}$) and simplify using the Greatest Common Factor (GCF).
- Decimals to Percentages:** Shift the decimal point two places to the right (multiply by 100) and add the % sign.

Step 1: Convert Decimal to Fraction

For 0.45, place 45 over 100 ($\frac{45}{100}$). Divide numerator and denominator by their GCF of 5 to get $\frac{9}{20}$.

Step 2: Convert Decimal to Percent

Multiply 0.45 by 100 to yield 45%.



Key Benchmark Equivalents

Memorizing standard benchmark equivalents allows rapid mental arithmetic and verification across mathematical operations.

- $\frac{1}{2} = 0.5 = 50\%$
- $\frac{1}{4} = 0.25 = 25\%$ and $\frac{3}{4} = 0.75 = 75\%$
- $\frac{1}{5} = 0.2 = 20\%$, $\frac{2}{5} = 0.4 = 40\%$, $\frac{3}{5} = 0.6 = 60\%$, $\frac{4}{5} = 0.8 = 80\%$
- $\frac{1}{10} = 0.1 = 10\%$ and $\frac{1}{100} = 0.01 = 1\%$
- $\frac{1}{3} = 0.333... = 33\frac{1}{3}\%$ and $\frac{2}{3} = 0.666... = 66\frac{2}{3}\%$
- $\frac{1}{8} = 0.125 = 12.5\%$ and $\frac{3}{8} = 0.375 = 37.5\%$



Benchmark Tip

Use benchmarks to estimate and sanity check answers. If a fraction is slightly greater than $\frac{1}{2}$, its percentage must be slightly greater than 50%.



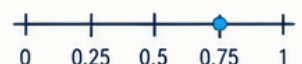
Fractions

To change any fraction into an equivalent decimal or percentage, perform division or scale to a power of ten.

- Fractions to Decimals:** Divide the numerator by the denominator using long division until the remainder terminates or repeats.
- Fractions to Percentages:** First convert to a decimal and multiply by 100%, or multiply the fraction directly by 100%.

Example: $\frac{3}{4}$

- Divide: $3 \div 4 = 0.75$
 - Convert to percent: $0.75 \times 100 = 75\%$
- So, $\frac{3}{4} = 0.75 = 75\%$.



$$\frac{3}{4} = 0.75 = 75\%$$

Unit 29: Ratios, Rates, and Equivalent Ratio Tables Mind Map

1



Core Definition & Representation

A ratio compares two quantities using multiplication and division. It expresses either a part-to-part relationship or a part-to-whole relationship across three standard notations.

- Word notation: "a to b" (e.g., 3 to 2)
- Colon notation: "a : b" (e.g., 3 : 2)
- Fractional notation: a/b (e.g., $\frac{3}{2}$)



Order Matters

The order of terms must match the stated comparison. If the ratio of boys to girls is 3:2, reversing it to 2:3 changes the meaning to girls to boys.

2



Part-to-Part vs. Part-to-Whole

Understanding whether a comparison relates subsets to each other or a subset to the entire collection is essential for setting up correct mathematical models.

- **Part-to-Part:** Compares one subset directly to another subset (e.g., 3 red marbles to 5 blue marbles, written 3:5).
- **Part-to-Whole:** Compares one subset to the combined total (e.g., 3 red marbles to 8 total marbles, written 3:8 or $\frac{3}{8}$).
- **Total Calculation:** Sum all distinct parts to establish the whole denominator ($3 + 5 = 8$).

3



Rates & Unit Rates

A rate is a ratio comparing two quantities with different units of measurement. A unit rate simplifies the comparison so that the second quantity equals 1 unit.

- **General Rate:** 120 miles driven in 3 hours = 120 miles / 3 hours
- **Unit Rate Calculation:** Divide both terms by the denominator value: $(120 / 3) / (3 / 3) = 40$ miles per 1 hour (40 mph)
- **Price per Unit:** Total cost divided by item count yields unit cost (e.g., \$6.00 for 4 apples = \$1.50 per apple)



40 mph

4



Equivalent Ratio Tables & Scaling

Equivalent ratios maintain an identical constant multiplicative relationship. A ratio table organizes scaled pairs using row-by-row or column-by-column multiplication.

- **Scale Factor Rule:** Multiply or divide both terms by the same non-zero number (c/c).
- Row 1: 2 cups concentrate to 5 cups water (Base ratio 2:5)
- Row 2: $(2 \times 2) = 4$ cups concentrate to $(5 \times 2) = 10$ cups water (Scale factor: 2)
- Row 3: $(2 \times 3) = 6$ cups concentrate to $(5 \times 3) = 15$ cups water (Scale factor: 3)
- Row 4: $(2 \times 6) = 12$ cups concentrate to $(5 \times 6) = 30$ cups water (Scale factor: 6)

Concentrate (cups)	Water (cups)
2	5
4	10
6	15
12	30



Multiplication vs. Addition Mistake

Never generate equivalent ratios by adding or subtracting the same number to both terms. Adding 2 to 2:5 gives 4:7, which does not equal 2:5.

5



Step-by-Step Worked Problem: Solving Proportions

Follow these structured procedural steps to find an unknown value in a proportional relationship using equivalent scaling.

1

Step 1: Identify Given Quantities

A recipe uses 3 cups of flour for every 4 tablespoons of butter. Determine how much butter is needed for 18 cups of flour.

2

Step 2: Set Up the Proportional Statement

Write the proportion:

$$\frac{3 \text{ cups flour}}{4 \text{ tbsp butter}} = \frac{18 \text{ cups flour}}{x \text{ tbsp butter}}$$

3

Step 3: Determine the Multiplicative Scale Factor

Compare matching known quantities: 18 divided by 3 equals a scale factor of 6.

4

Step 4: Scale the Dependent Quantity

Multiply the denominator term by the identical scale factor: $4 \times 6 = 24$. Thus, 24 tablespoons of butter are required.

RATIOS, RATES & TABLES

6



Verification Checklist & Diagnostic Practice

Apply this systematic check before finalizing any proportional problem or table calculation.

- Verify whether the problem asks for part-to-part or part-to-whole before calculating fractions.
- Check that both terms in the ratio were multiplied or divided by the identical non-zero factor.
- Confirm units are preserved correctly across numerators and denominators.
- Test simplified fractions: verify that simplified $(\frac{18}{72})$ reduces directly back to $(\frac{3}{4})$.



Diagnostic Self-Check

If cross-multiplying $(3 \times 24 = 72 \text{ and } 4 \times 18 = 72)$, the cross products must be strictly equal.

Base Ratio Recorded (a:b):

Target Ratio Setup:

Multiplier / Scale Factor Used:

Final Solution with Units:



Are the units in the comparison identical or distinct?

Identical Units
(e.g., cups to cups, students to students)

Express as a standard unitless ratio
(e.g., 3:2 or $\frac{3}{2}$).

Different Units
(e.g., miles to hours, dollars to items)

Express as a rate or unit rate with compound units
(e.g., 40 miles/hour or \$1.50/apple).

Unit Rates, Constant Speed, and Best Buy Comparisons



Understanding Unit Rates

A rate compares two different quantities with distinct measurement units. A unit rate simplifies this comparison so that the quantity in the denominator equals exactly 1.

- **Rate Definition:** A ratio comparing quantities with different physical or monetary units (such as miles to hours or dollars to ounces).
- **Unit Rate Form:** Written as a fraction a/b where the denominator b is reduced to 1 through division.
- **Notation:** Read with the word 'per', signifying for each single standard unit (e.g., miles per hour, price per ounce).



Unit Rate Rule

To find any unit rate, divide the top number (numerator) by the bottom number (denominator). The resulting denominator is always 1.

Steps to Find a Unit Rate

- 1 **Identify the Given Rate**
Write the given ratio with quantities and their respective units, such as 150 miles / 3 hours.
- 2 **Divide by the Denominator**
Divide both the numerator and the denominator by the denominator value: $(150 / 3) / (3 / 3)$.
- 3 **State the Unit Value**
Express the simplified result: 50 miles / 1 hour, or 50 miles per hour (mph).



Constant Speed and Motion

Constant speed is a specific unit rate relating total distance traveled to total elapsed time under uniform motion.

- **Distance Formula:** Distance = Speed $\times$ Time ($d = s \times t$).
- **Speed Formula:** Speed = Distance / Time ($s = d / t$).
- **Time Formula:** Time = Distance / Speed ($t = d / s$).



Worked Speed Example

A cyclist travels 36 miles in 3 hours. Divide distance by time: 36 miles / 3 hours = 12 miles per hour.

Which variable is unknown in your motion problem?

Speed is unknown



Divide total distance by total time: $s = d / t$.

Distance is unknown



Multiply constant speed by total time: $d = s \times t$.

Time is unknown



Divide total distance by constant speed: $t = d / s$.

Unit Rates & Best Buys



Unit Price and Best Buy Comparisons

Unit pricing allows consumers to objectively compare items packaged in different quantities by calculating the cost per single standard unit.

- **Unit Price Formula:** Total Cost / Total Units = Price per Unit.
- **Standardized Units:** Convert package quantities to identical units such as fluid ounces, grams, or pounds before dividing.
- **Decision Criterion:** The item with the lowest unit price provides the greatest quantity per dollar, representing the best buy.



Common Mistake: Inverting the Ratio

Do not divide ounces by dollars. Unit price requires Cost divided by Quantity (\$ / unit), yielding dollars per ounce, not ounces per dollar.

Steps to Find the Best Buy

- 1 **Standardize Measurement Units**
Ensure both competing items use the same unit of measure (e.g., convert pounds to ounces if comparing against an ounce container).
- 2 **Calculate First Unit Price**
Divide total package price by quantity: Brand A costs \$4.80 for 16 oz = $\$4.80 / 16 = \0.30 per oz.
- 3 **Calculate Second Unit Price**
Divide competitor price by quantity: Brand B costs \$6.00 for 24 oz = $\$6.00 / 24 = \0.25 per oz.
- 4 **Compare Values**
Identify the lower cost per ounce: \$0.25 per oz is less than \$0.30 per oz, making Brand B the best buy.



Tabular Rate Comparisons & Practice

Tabular rate representations organize multiples and unit values into aligned rows, making proportionality and comparison visible at a glance.

- **Proportional Scaling:** Multiply or divide both columns by identical factors.
- **Check Denominator:** Confirm the row corresponding to 1 unit contains the verified unit rate.

Item	Total Price	Package Quantity	Unit Price (\$ / unit)
A	\$4.80	16 oz	\$0.30
B	\$6.00	24 oz	\$0.25



Your Turn

Item A: Total Price and Package Quantity

Item A: Calculated Unit Price (\$ / unit)

Item B: Total Price and Package Quantity

Item B: Calculated Unit Price (\$ / unit)

Identified Best Buy and Mathematical Reason



Unit 31: Proportions & Cross-Multiplication Mind Map



1. DEFINITION & ANATOMY

Understanding Proportions

A proportion is a mathematical statement showing that two ratios or fractions are equivalent. In standard fractional notation, a proportion is expressed as $a/b = c/d$, where $b \neq 0$ and $d \neq 0$.

$$\frac{a}{b} = \frac{c}{d}$$

- **Ratio equivalence:** $a/b = c/d$ represents two balanced comparative rates.
- **Terms:** In the proportion $a/b = c/d$, "a" and "d" are termed the extremes, while "b" and "c" are termed the means.
- **Equality condition:** The proportion holds true if and only if the decimal or simplified fractional values of both sides are strictly identical.



Fundamental Property of Proportions

For any valid proportion $\frac{a}{b} = \frac{c}{d}$, multiplying both sides by the common denominator ($b \cdot d$) yields the cross-product identity: $a \cdot d = b \cdot c$.

PROPORTIONS & CROSS-MULTIPLICATION



2. SOLVING PROCEDURE

When solving an algebraic proportion containing an unknown variable, cross-multiplication isolates the unknown by converting fractional equations into linear equations.

- **Identify** the unknown variable position within numerator or denominator.
- **Multiply** opposite diagonals to equate products.
- Use division properties of equality to solve for the missing term.

1 Step 1: Set Up the Proportional Equation

Write the proportion clearly with known values and the unknown variable, ensuring consistent units:

$$\frac{x}{12} = \frac{5}{15}$$

2 Step 2: Calculate the Diagonal Cross-Products

Multiply numerator of each side by the denominator of the opposite side: $x \cdot 15 = 12 \cdot 5$.

3 Step 3: Simplify the Resulting Arithmetic

Calculate the numerical product: $15x = 60$.

4 Step 4: Isolate the Variable

Divide both sides by the coefficient 15:

$$x = \frac{60}{15}, \text{ yielding the solution } x = 4.$$

5 Step 5: Verify the Equality

Substitute $x = 4$ back into the initial fraction:

$$\frac{4}{12} \text{ simplifies to } \frac{1}{3}, \text{ and } \frac{5}{15} \text{ simplifies to } \frac{1}{3}.$$

The equivalence confirms the solution.



4. COMMON MISTAKE ALERT

Never multiply straight across (numerator by numerator, denominator by denominator) when solving a proportion. Straight-across multiplication applies to fraction multiplication, not equations separated by an equals sign.

$$\frac{a}{b} = \frac{c}{d} \quad \times$$

Do NOT multiply straight across!



3. WORKED EXAMPLES & VERIFICATION

Examine worked examples for both whole number and decimal scenarios to reinforce cross-product consistency.

- **Example A (Direct solve):**

$$\frac{3}{8} = \frac{y}{24} \rightarrow 3y = 72 \rightarrow y = 24.$$

Check: $\frac{3 \cdot 6}{8 \cdot 3} = 0.375$; $\frac{y \cdot 3}{24 \cdot 3} = 0.375$.

- **Example B (Fractional rate):**

$$\frac{2.5}{5} = \frac{k}{14} \rightarrow 5k = 35 \rightarrow k = 7.$$

Check: $\frac{2.5 \cdot 5}{2 \cdot 5} = 0.5$; $\frac{7 \cdot 5}{7 \cdot 14} = 0.5$.



Common Mistake Alert

Never multiply straight across (numerator by numerator, denominator by denominator) when solving a proportion. Straight-across multiplication applies to fraction multiplication, not equations separated by an equals sign.



5. PROPORTIONAL SETUP & PRACTICE RECORD

Use this structured template to set up, cross-multiply, and verify unknown values in applied problem sets.

- Ensure matching dimensional units appear across corresponding positions.
- Record cross-products explicitly before dividing.



Facilitator Tip

Encourage students to draw diagonal loops around cross-terms to visually pair extremes and means before writing the algebraic line.

Original Proportion Equation
(e.g., $a/b = c/d$)

Diagonal Cross-Product Line
($a \cdot d = b \cdot c$)

Arithmetic Simplification and Isolation Step

Final Solution and Verification Check



Does the unknown appear in the numerator or the denominator?

Unknown is in the numerator

$$\frac{a/b}{c/d} = \frac{c/d}{c/d}$$

Cross-multiply directly to get $dx = bc$, then divide both sides by d.

$$x = \frac{bcd}{d}$$

Unknown is in the denominator

$$\frac{a/b}{c/d} = \frac{c/d}{c/d}$$

Cross-multiply to get $cx = ad$, then divide both sides by c.

$$x = \frac{ad}{c}$$



Percent Foundations: Modeling Parts per Hundred



Core Definition: Parts per Hundred

The word percent originates from the Latin *per centum*, meaning "by the hundred". A percent represents a ratio or fraction whose denominator is fixed at 100. Any whole can be visualized as 100 equal pieces, where each piece represents exactly 1 percent (1%).

- Symbol: % indicates parts out of one hundred
- Fraction representation: $P\% = P / 100$
- A complete single whole equals 100% ($100 / 100 = 1$)



Base 100 Anchor

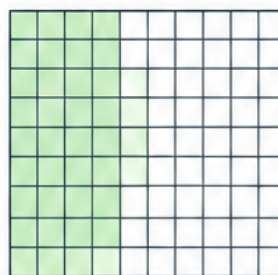
Always write the given percent over 100 before simplifying. For example, 45% is simply $45/100$, which reduces to $9/20$.



Visual Shading Models: The 10x10 Grid

A standard 10 by 10 grid contains 100 identical small squares. Shading squares directly maps the percentage value into a geometric area.

- 1 full column or row of 10 squares equals 10%
- Shading 25 squares covers one quarter of the grid (25%)
- Shading 50 squares covers half of the grid (50%)
- Shading all 100 squares covers the entire unit area (100%)



- 1 Identify Total and Target
Determine the percentage needed, such as 35%.
- 2 Fill Complete Columns
Color 3 full 10-square columns to represent 30 squares.
- 3 Fill Remaining Units
Color 5 additional individual squares to reach 35 shaded squares out of 100.

Percent Foundations



Mental Math Percent Benchmarks

Key benchmark percentages allow fast mental calculations without manual long division or scratch paper.

- 50% Benchmark: Divide the total quantity by 2 (50% of 80 = 40)
- 25% Benchmark: Divide the total quantity by 4, or halve 50% (25% of 80 = 20)
- 10% Benchmark: Move the decimal point one place to the left, dividing by 10 (10% of 80 = 8)
- 5% Benchmark: Take half of the 10% value (5% of 80 = 4)



Worked Numeric Example

Calculate 35% of 60 using benchmarks:
10% of 60 is 6. Therefore, 30% is $3 \times 6 = 18$.
Half of 10% is 5%, which equals 3.
Summing 30% and 5%: $18 + 3 = 21$.
Thus, 35% of 60 = 21.



Practice problem: Find 15% of 120 using mental benchmarks



Grid shading: How many squares out of 100 represent 62%?



Direct Conversions: Percent to Decimal

Converting a percentage to a decimal relies on dividing by 100. In base-10 arithmetic, dividing by 100 shifts the decimal point exactly two places to the left.

- $75\% = 75.0 / 100 = 0.75$
- $8\% = 08.0 / 100 = 0.08$ (insert a leading zero)
- $125\% = 125.0 / 100 = 1.25$ (values over 100% exceed 1)



Common Mistake

Do not write 6% as 0.6. Moving the decimal two places left requires a placeholder zero:
 $6\% = 0.06$. Note that 0.6 represents 60%.



Is the percent a single digit or multiple digits?

Single digit
(e.g., 4%)

Double digit or larger
(e.g., 64%)

Add a leading zero
before shifting:
4% becomes 0.04.

Shift the decimal point
two places directly to the left:
64% becomes 0.64.



Basic Math Mind Maps: Solving Percent Problems



Percent Relationship Hub: The Core Formulas

Percent problems compare a portion of a quantity to the entire total. Every percent situation connects three fundamental components: the Part (A), the Whole or Base (B), and the Percent rate (P). Two standard algebraic representations allow you to solve for any missing term.

- Percent Proportion: $\frac{\text{Part}}{\text{Whole}} = \frac{\text{Percent}}{100}$
- Percent Equation: $\text{Part} = \frac{\text{Percent}}{100} \times \text{Whole}$,
or $\text{Part} = \text{Rate} \times \text{Whole}$
- Part (A): The specific share, subset, or change being measured (often paired with 'is').
- Whole (B): The total baseline quantity representing 100% (often preceded by 'of').
- Percent (P%): The ratio expressed as a amount per 100.



Equation vs. Proportion

When using the equation **Part = Rate × Whole**, always convert the percentage into a decimal first (e.g., 25% = 0.25).

When using the proportion **Part / Whole = P / 100**, keep P as the integer percentage value.



Finding the Percent ($P = \text{Part} / \text{Whole} \times 100$)

Use this pathway when both the part and the whole are given, and you need to compute the corresponding rate per hundred.

- Formula Method:
 $P = (\text{Part} / \text{Whole}) \times 100\%$
- Proportion Method:
Solve for P in $\frac{\text{Part}}{\text{Whole}} = \frac{P}{100}$

Worked Example: 18 is what percent of 72?

1. Identify the values: Part = 18, Whole = 72, Percent P is unknown.
2. Set up the equation: $18 = (\frac{P}{100}) \times 72$.
3. Divide 18 by 72: $\frac{18}{72} = 0.25$.
4. Convert to percentage: $0.25 \times 100 = 25\%$.
5. Conclusion: 18 is 25% of 72.

Verification via Proportion

Set up $\frac{18}{72} = \frac{P}{100}$.
Reduce $\frac{18}{72}$ to $\frac{1}{4}$: $\frac{1}{4} = \frac{P}{100}$.
Cross-multiply: $4 \times P = 100$, so $P = 25\%$.

Solving Percent Problems: Part, Whole, & Percent



Finding the Whole ($\text{Whole} = \text{Part} / (P / 100)$)

Use this pathway when a known portion (part) represents a given percentage, and you must determine the total original quantity (whole).

- Formula Method:
 $\text{Whole} = \text{Part} / (P / 100)$
- Proportion Method:
Cross-multiply $\frac{\text{Part}}{\text{Whole}} = \frac{P}{100}$
to yield $\text{Whole} = (\text{Part} \times 100) / P$

Worked Example: 45 is 60% of what number?

1. Identify the values: Part = 45, Percent P = 60, Whole is unknown (w).
2. Set up the equation: $45 = 0.60 \times w$.
3. Isolate w by dividing: $w = 45 / 0.60 = 75$.
4. Conclusion: 45 is 60% of 75.

Verification via Proportion

Set up $\frac{45}{w} = \frac{60}{100}$.
Cross-multiply: $60 \times w = 45 \times 100 = 4500$.
Divide by 60: $w = 4500 / 60 = 75$.



Finding the Part ($\text{Part} = \% \times \text{Whole}$)

Use this pathway when the total whole and the percentage rate are known, and you need to find the specific portion amount.

- Formula Method:
 $\text{Part} = (\frac{P}{100}) \times \text{Whole}$
- Proportion Method:
Cross-multiply $\frac{\text{Part}}{\text{Whole}} = \frac{P}{100}$
to yield $\text{Part} = (\text{Whole} \times P) / 100$

Worked Example: What is 35% of 80?

1. Identify the values: Percent P = 35, Whole = 80, Part is unknown (x).
2. Formulate the equation: $x = (\frac{35}{100}) \times 80$.
3. Simplify: $x = 0.35 \times 80 = 28$.
4. Conclusion: 28 is 35% of 80.

Verification via Proportion

Set up $\frac{x}{80} = \frac{35}{100}$.
Cross-multiply: $100 \times x = 80 \times 35 = 2800$.
Divide both sides by 100: $x = 28$.



Common Mistakes

- Forgetting to convert percent to a decimal (e.g., 25% = 0.25).
- Multiplying by 25 instead of 0.25 (inflates the part by 100x).
- Confusing part and whole (the number after 'of' is usually the whole).

Remember:

Never divide by 100 twice!

If you convert a percent to a decimal (0.15), do NOT write it over 100 in the proportion formula.

Real-World Percent Applications Mind Map



Real-World Percent Fundamentals

Everyday financial math uses percentages to calculate increases and decreases on a base amount. Converting percentages to decimals by dividing by 100 allows direct calculation of tax, tips, markups, and discounts.



- Percent means parts per one hundred: divide by 100 to convert to decimal form (e.g., $7.5\% = 0.075$).
- Base Value represents 100% of the starting cost before additions or deductions.
- Additions increase the total cost: $\text{Total} = \text{Base} + \text{Amount Added}$.
- Subtractions decrease the total cost: $\text{Total} = \text{Base} - \text{Amount Deducted}$.



Core Multiplier Rule

Percent increase multiplier equals $1 + r$.
Percent decrease multiplier equals $1 - r$,
where r is the decimal rate.



Calculating Sales Tax and Gratuity

Sales tax and restaurant tips are proportional additions calculated directly from the original subtotal.



- $\text{Tax Amount} = \text{Subtotal} \times \text{Tax Rate}$.
- $\text{Tip Amount} = \text{Subtotal} \times \text{Tip Rate}$.
- $\text{Total Bill} = \text{Subtotal} + \text{Tax} + \text{Tip}$.
- Shortcut method: Multiply subtotal directly by $(1 + \text{combined decimal rate})$.



Common Mistake: Tipping on the Total

Do not compute gratuity on top of sales tax unless explicitly instructed. Gratuity is typically assessed on the food and beverage subtotal.

Steps:

- 1 **Convert Rate to Decimal**
Express the tax rate as a decimal (e.g., 8.25% becomes 0.0825).
- 2 **Multiply by Base Price**
Multiply the original subtotal by the decimal tax rate to find the tax in dollars.
- 3 **Add to Base Price**
Add the calculated tax amount to the original subtotal to find the final price.



Discounts and Markups

Discounts lower the original price during sales, while markups raise wholesale costs to retail prices.

- $\text{Discount Amount} = \text{Original Price} \times \text{Discount Rate}$.
- $\text{Sale Price} = \text{Original Price} - \text{Discount Amount} = \text{Original Price} \times (1 - \text{Discount Rate})$.
- $\text{Markup Amount} = \text{Wholesale Cost} \times \text{Markup Rate}$.
- $\text{Retail Price} = \text{Wholesale Cost} + \text{Markup Amount} = \text{Wholesale Cost} \times (1 + \text{Markup Rate})$.



Common Mistake: Inverting Single Multipliers

A 25% discount means you pay 75% (0.75) of original price, not 0.25 . A 40% markup means the customer pays 140% (1.40) of wholesale.



Are you computing an increase or a decrease on the base price?

Increase (Tax, Tip, Markup)

Add the percentage to 100% and multiply:
 $\text{Final} = \text{Base} \times (1 + \text{rate})$.

Decrease (Discount, Sale, Markdown)

Subtract the percentage from 100% and multiply:
 $\text{Final} = \text{Base} \times (1 - \text{rate})$.



Student Practice & Self-Check

Solve each problem using either two-step calculation or single-multiplier shortcut methods.

- Compute the sales tax on a \$120.00 tablet at 7% tax rate.
- Determine the final bill for a \$54.00 restaurant tab with an 18% tip.
- Find the sale price of a \$95.00 backpack discounted by 15%.



Problem 1: Final price of \$120 tablet at 7% sales tax

Problem 2: Total cost of \$54 bill with 18% tip

Problem 3: Final price of \$95 backpack at 15% discount

Real-World Percent Applications



Worked Numeric Examples

Verify everyday calculations step by step using standard multi-step and single-multiplier methods.

- Example 1: A jacket costs \$84.00 with a 30% discount. Sale price = $\$84.00 \times (1 - 0.30) = \$84.00 \times 0.70 = \$58.80$.
- Example 2: A store purchases a pair of shoes at wholesale for \$45.00 and marks it up by 60%. Retail price = $\$45.00 \times (1 + 0.60) = \$45.00 \times 1.60 = \$72.00$.
- Example 3: Dinner subtotal is \$65.00. Tax is 8% (\$5.20) and tip is 20% (\$13.00). Total payable = $\$65.00 + \$5.20 + \$13.00 = \83.20 .



Percent of Change Mind Map



Core Principles of Percent Change

Percent of change measures relative change compared to the original starting value. The fundamental relationship compares the absolute difference between values against the initial baseline.

Formula:

$$\text{Percent Change} = \frac{|\text{New Value} - \text{Old Value}|}{\text{Old Value}} \times 100\%$$

- The denominator is ALWAYS the original starting baseline (Old Value), never the new value.
- Change is classified as an Increase when New Value > Old Value.
- Change is classified as a Decrease when New Value < Old Value.



Denominator Trap

Never divide by the new value when calculating standard percent change. Always use the original starting amount as the divisor.



Worked Numeric Examples

Follow step-by-step arithmetic procedures for computing percent increase, percent decrease, and laboratory percent error.

- **Percent Increase:**
Old = 40, New = 50.
Change = $50 - 40 = 10$.
Rate = $(10 / 40) \times 100\% = 25\%$ increase.
- **Percent Decrease:**
Old = 80, New = 60.
Change = $80 - 60 = 20$.
Rate = $(20 / 80) \times 100\% = 25\%$ decrease.
- **Percent Error:**
Theoretical = 120, Experimental = 114.
Difference = $|114 - 120| = 6$.
Error = $(6 / 120) \times 100\% = 5\%$ error.



Theoretical Baseline

In percent error calculations, the theoretical (accepted standard) value serves as the denominator, analogous to the old value.



Self-Check Practice and Review

Apply the relative change procedure to practice problems to verify accuracy before proceeding to advanced applications.

- Item 1: A price moves from \$25 to \$30. State the percentage change.
- Item 2: Enrollment falls from 200 students to 170 students. State the percentage change.
- Item 3: A measurement yields 9.8 m/s^2 against an accepted target of 10.0 m/s^2 . State the percent error.



Verification Note

Ensure units match before taking differences, and keep all intermediate division decimals unrounded until the final percentage step.

Problem 1 Calculation & Answer

Problem 2 Calculation & Answer

Problem 3 Calculation & Answer

Percent of Change



How do you classify and calculate relative change between two measured values?

How do you classify and calculate relative change between two measured values?

New Value is greater than Old Value

Subtract Old from New, divide by Old, and multiply by 100 to find the Percent Increase.

New Value is less than Old Value

Subtract New from Old, divide by Old, and multiply by 100 to find the Percent Decrease.



Common Mistake

Do not divide by the new value when calculating percent change. Always use the original starting value as the divisor.



Simple Interest Formula & Principal Accrual Map



Core Equation & Variable Breakdown

Simple interest calculates the financial return or borrowing cost applied solely to the original principal amount across time.

- **I = Simple Interest:** Total dollar amount of interest earned or paid.
- **P = Principal:** Initial sum of money deposited, borrowed, or invested.
- **r = Annual Interest Rate:** Yearly percentage expressed in decimal form (for example, $5.5\% = 0.055$).
- **t = Time:** Duration of the loan or investment measured in years (fractions of a year convert to months divided by 12).



Percent to Decimal Rule

Always divide the stated percentage rate by 100 before multiplying. Never insert whole percentages directly into the formula.



Step-by-Step Calculation

- 1 **Identify Given Values**
Extract principal (P), annual interest rate (r), and total duration in years (t) directly from the problem statement.
- 2 **Convert Rate to Decimal**
Divide the annual rate percentage by 100 (drop the % sign) to express r as a standard decimal value.
- 3 **Multiply to Find Interest**
Compute $I = P \cdot r \cdot t$ to determine the total simple interest accrued across the entire investment period.
- 4 **Calculate Total Balance**
Sum the initial principal and accrued interest using $A = P + I$ (or $A = P(1 + rt)$) to establish total future value.



Worked Numeric Examples

Step-by-step verified numerical applications demonstrating linear interest accrual over single and multi-year durations.

Example 1:

A \$3,800 loan at an annual rate of 5.5% for 5 years yields
 $I = \$3,800 \cdot 0.055 \cdot 5 = \$1,045$
in interest, yielding a total balance of \$4,845.

Example 2:

A \$1,200 deposit at an annual rate of 4% for 18 months (1.5 years) yields
 $I = \$1,200 \cdot 0.04 \cdot 1.5 = \72
in interest, yielding a total balance of \$1,272.



Common Mistake: Time Unit Mismatch

If time is provided in months or days, it must be converted to years before calculating. For instance, 6 months equals $6/12 = 0.5$ years.

Simple Interest:
$$I = P \cdot r \cdot t$$



Which variable is unknown in your simple interest problem?

Interest earned (I) is unknown

Multiply principal, annual rate as a decimal, and time in years directly:

$$I = P \cdot r \cdot t$$

Total balance accrued (A) is unknown

Compute interest first with $I = P \cdot r \cdot t$, then add to principal:

$$A = P + I$$

Principal (P), rate (r), or time (t) is unknown

Isolate the unknown variable algebraically:

$$P = \frac{I}{(r \cdot t)}$$
$$r = \frac{I}{(P \cdot t)}$$
$$t = \frac{I}{(P \cdot r)}$$



Integer Concepts, Absolute Value, and Opposite Numbers Mind Map



Integers and the Horizontal Number Line

The set of integers includes all positive counting numbers, their negative opposites, and zero: {..., -3, -2, -1, 0, 1, 2, 3, ...}. On a horizontal number line, zero acts as the neutral reference origin with no sign. Positive integers lie to the right of zero, increasing in magnitude as you move rightward. Negative integers lie to the left of zero, decreasing in value as you move leftward.

- Positive integers (+1, +2, +3, ...) indicate values greater than zero.
- Zero (0) is neither positive nor negative.
- Negative integers (-1, -2, -3, ...) indicate values less than zero.
- Any number to the right on a horizontal number line is strictly greater than any number to its left.



Direction and Value

Moving right always means increasing value; moving left always means decreasing value. For example, -2 is to the right of -5, so $-2 > -5$.



Opposite Numbers (Additive Inverses)

Two numbers are opposites if they are the exact same distance from zero on the number line, but lie in opposite directions. The opposite of a positive integer is its corresponding negative integer, and the opposite of a negative integer is positive. Zero is its own opposite.

- The opposite of a is $-a$.
- The opposite of $-a$ is $-(-a) = a$.
- The sum of any number and its opposite always equals zero: $a + (-a) = 0$.



Common Mistake

Do not assume the negative symbol '-' always makes an answer negative. The opposite of a negative number is positive: $-(-7) = 7$.



Absolute Value as Distance

The absolute value of a real number, written with vertical bars as $|x|$, measures its non-directional distance from zero along the number line. Because distance can never be negative, absolute value is always non-negative: $|x| \geq 0$ for all integers x .

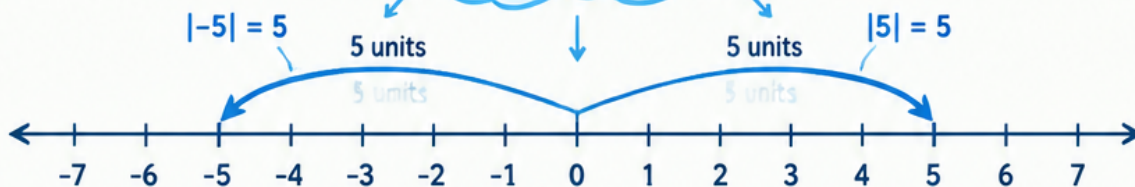
- If $a > 0$, then $|a| = a$ (example: $|6| = 6$).
- If $a = 0$, then $|0| = 0$.
- If $a < 0$, then $|a| = -a$ (example: $|-6| = -(-6) = 6$).



Distance is Never Negative

An absolute value sign never produces a negative result. Operations outside the bars, however, are applied afterward: $-|-4| = -(4) = -4$.

INTEGERS & ABSOLUTE VALUE



Comparing & Ordering Integers

Follow these direct procedural steps to compare or order any collection of signed integers.

1

Plot Values on the Number Line

Locate zero, then mark every given positive integer to the right and every negative integer to the left.

2

Evaluate Enclosed Expressions

Simplify any absolute value or double-negative expressions first (e.g., convert $|-8|$ to 8 and $-(-3)$ to 3) before comparing. Read positions from left to right. Place the numbers in order from far left to far right. The number farthest left is the least.

4

Apply Inequality Symbols

Use the less-than symbol ($<$) or greater-than symbol ($>$) pointing towards the smaller quantity (e.g., $-9 < -4 < 0 < 2$).



Worked Numeric Examples and Practice Verification

Review these fully worked standard solutions demonstrating comparison and absolute value evaluation.

- Example 1: Compare -7 and -3. On the number line, -7 is to the left of -3, so $-7 < -3$.
- Example 2: Evaluate $|-15| - |9|$. $|-15| = 15$ and $|9| = 9$, so $15 - 9 = 6$.
- Example 3: Solve $|x| = 8$. Since both 8 and -8 are 8 units away from 0, $x = 8$ or $x = -8$.
- Example 4: Order $\{-4, 2, -6, 0, |-5|\}$ from least to greatest. First simplify $|-5| = 5$. Left-to-right order: -6, -4, 0, 2, 5.



Practice

- Find the opposite of -12 and graph both values on a line.
- Evaluate the expression $-|-18| + 5$.
- Order the set $\{-11, 4, -2, |-7|, -8\}$ from least to greatest.

Mind Map Unit 38:

Adding Integers with Same and Different Signs



Rule 1: Adding Integers with the Same Sign

When adding two numbers that have the same sign (both positive or both negative), add their absolute values together and keep the common sign.

Visualizing with counters: two negative counters combine with three negative counters to make five negative counters.



- Positive + Positive = Positive sum: $(+4) + (+2) = +6$
- Negative + Negative = Negative sum: $(-3) + (-5) = -8$
- Number line jump: Start at the first number, then move in the direction of the second number (left for negative, right for positive).



Common Mistake

Do not confuse addition rules with multiplication. Two negative numbers added together do not produce a positive result. Example: $(-4) + (-3) = -7$, NOT $+7$.

Steps

- Step 1: Identify Signs**
Check both addends. Determine whether both numbers share identical positive or negative signs.
- Step 2: Add Absolute Values**
Ignore the signs temporarily and calculate the sum of their numerical magnitudes.
- Step 3: Attach the Shared Sign**
Apply the original shared sign to the resulting sum.

Practice: Calculate

$$(-6) + (-9) =$$



Rule 2: Adding Integers with Different Signs

When adding two numbers with different signs (one positive and one negative), subtract the smaller absolute value from the larger absolute value. The final sum takes the sign of the number with the larger absolute value.

Counter model: one positive counter and one negative counter cancel out to form a zero pair.



- Find $|a|$ and $|b|$ to identify which value is further from zero.
- Subtract the smaller absolute value from the larger: $|larger| - |smaller|$.
- Assign the sign of the addend with the larger absolute value.
- **Example 1:** $(-7) + (+4) \rightarrow 7 - 4 = 3 \rightarrow$ Sign of -7 dominates $\rightarrow$ Result = -3
- **Example 2:** $(+8) + (-3) \rightarrow 8 - 3 = 5 \rightarrow$ Sign of $+8$ dominates $\rightarrow$ Result = $+5$

Zero Pair Principle

Any number added to its opposite equals zero: $(+a) + (-a) = 0$. When canceling zero pairs, what remains determines both the numeric value and sign of the answer.

Adding Integers

Which addend has the greater absolute value?

The positive number has the greater absolute value ($|pos| > |neg|$)

Subtract $|neg|$ from $|pos|$.
The final answer is positive.

The negative number has the greater absolute value ($|neg| > |pos|$)

Subtract $|pos|$ from $|neg|$.
The final answer is negative.

Both numbers have equal absolute values ($|pos| = |neg|$)

The numbers cancel out completely into zero pairs. The result is 0.



Worked Examples and Verification Checklist

Review these fully verified step-by-step solutions to validate procedure accuracy using both number line models and arithmetic rules.

- **Problem A:** $(-12) + (-8) = -20$ (Same sign: $12 + 8 = 20$, retain negative)
- **Problem B:** $(+15) + (-6) = +9$ (Different signs: $15 - 6 = 9$, retain positive from $+15$)
- **Problem C:** $(-14) + (+9) = -5$ (Different signs: $14 - 9 = 5$, retain negative from -14)
- **Problem D:** $(-11) + (+11) = 0$ (Opposite values cancel to form zero)



Self-Check Strategy

Verify the final sign before finishing: if you started negative and added a smaller positive, you must still be to the left of zero on the number line.



Guided Problem: $(-18) + (+7) =$ _____

Guided Problem: $(+13) + (-25) =$ _____

Mind Map 39: Subtracting Integers via Additive Inverses



Core Principle: Add the Opposite

Subtraction of any integer is mathematically defined as adding its additive inverse (opposite). Every subtraction problem can be rewritten into an addition problem without changing its numerical value.

- **Algebraic Definition:** $a - b = a + (-b)$
- **Additive Inverse Identity:** $b + (-b) = 0$
- **Subtracting a Positive:** $5 - 8$ becomes $5 + (-8) = -3$
- **Subtracting a Negative:** $4 - (-3)$ becomes $4 + 3 = 7$



Keep-Change-Change Rule

Keep the first integer exactly as it is, change the subtraction operator ($-$) to an addition operator ($+$), and change the sign of the subtrahend to its opposite.

1 Step 1: Identify Components

Distinguish the minuend (first term) and the subtrahend (term being subtracted). For example, in $-6 - (-9)$, the minuend is -6 and the subtrahend is -9 .

2 Step 2: Apply the Additive Inverse

Rewrite the subtraction sign as addition, and flip the sign of the subtrahend: $-6 - (-9)$ becomes $-6 + (+9)$.

3 Step 3: Execute Standard Addition Rules

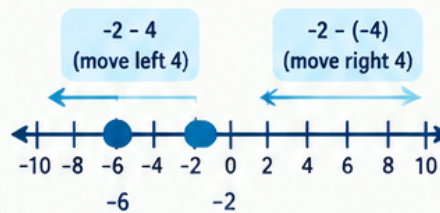
Add the signed terms: since -6 and $+9$ have different signs, find the absolute difference ($9 - 6 = 3$) and apply the sign of the larger absolute value, yielding $+3$.



Visual Vector Directions on the Number Line

Number line vectors clarify the geometric orientation of signed subtraction. Positive addition points toward the right (positive direction), while negative addition points toward the left (negative direction).

- **Start Point:** Place your initial marker on the number line at the minuend (a).
- **Subtracting Positive ($+b$):** Moving in reverse of positive means heading b units LEFT.
- **Subtracting Negative ($-b$):** Moving in reverse of negative means heading b units RIGHT.
- **Equivalence Check:** $-2 - 4$ lands at -6 ; $-2 + (-4)$ also lands at -6 .



Common Mistake: Subtracting Absolute Values Blindly

Do not simply compute $8 - 3$ when solving $-3 - 8$. In $-3 - 8$, rewriting yields $-3 + (-8) = -11$, moving further into negative territory rather than subtracting magnitude.



Multi-Term Expressions

When evaluating complex multi-step integer expressions with nested signs and multiple subtractions, rewrite all subtraction operations to additions from left to right before grouping terms.

- 1 **Expression:** $-12 - (-7) - 5$
- 2 **Rewrite 1:** $-12 + (+7) - 5$
- 3 **Rewrite 2:** $-12 + 7 + (-5)$
- 4 **Combine Negatives:** $(-12 + -5) + 7 = -17 + 7$
- 5 **Final Evaluated Solution:** -10



Grouping Strategy

Once all subtractions are transformed into additions of opposites, use the Commutative and Associative Properties to sum all negative integers together and all positive integers together before computing the final sum.



Problem 1: Evaluate $-15 - (+9)$ by rewriting as addition

Problem 2: Evaluate $8 - (-14)$ on a mental number line

Problem 3: Simplify $-4 - 11 - (-6)$ showing both rewrite steps

SUBTRACTING INTEGERS

$$a - b = a + (-b)$$



Decision: What sign does the subtrahend (second number) have?

Subtrahend is positive: $a - (+b)$

Rewrite as $a + (-b)$.
Move left along the number line by b units.

Subtrahend is negative: $a - (-b)$

Rewrite as $a + (+b)$.
Two consecutive negatives form an addition; move right along the number line by b units.



Multiplying and Dividing Signed Integers – Mind Map Guide



Two-Term Sign Rules

Multiplying and dividing signed integers follows consistent mathematical laws based on whether the signs of the two numbers are identical or opposite.

Positive × Positive = Positive	$6 \times 4 = 24$
Negative × Negative = Positive	$(-7) \times (-6) = 42$
Positive × Negative = Negative	$8 \times (-3) = -24$
Negative × Positive = Negative	$(-3) \times 5 = -15$
Same sign quotient	$(-24) \div (-8) = 3$
Opposite sign quotient	$(-30) \div 5 = -6$



Universal Two-Term Rule

For any two real numbers a and b :
 $(-a) \times (-b) = ab$, and $(-a) \times b = -(ab)$.
 Similarly, $(-a) \div (-b) = \frac{a}{b}$, and $(-a) \div b = -(a/b)$ where $b \neq 0$.



Multi-Factor Sign Counting Method

When multiplying three or more signed integers in a single expression, determine the sign of the overall product by counting the total quantity of negative factors. Because every pair of negative signs cancels out to form a positive factor, only the parity of negative terms dictates the ultimate sign.

- **Even number of negative factors** produces a **Positive** result:
 $(-2) \times (-3) \times (-4) \times (-5) = +120$
 (4 negatives)
- **Odd number of negative factors** produces a **Negative** result:
 $(-2) \times (-3) \times (-4) = -24$
 (3 negatives)
- Positive factors do not alter the final sign parity:
 $(-5) \times 2 \times (-3) = +30$ (2 negatives)
- Any expression containing a factor of 0 evaluates immediately to 0:
 $(-9) \times 4 \times 0 \times (-2) = 0$



Even / Odd Sign Shortcut

Count the '-' signs: An even count always yields '+'. An odd count always yields '-'. Positive factors have zero effect on sign determination.



What is the total count of negative factors in the multiplication chain?

The count of negative factors is an even number (0, 2, 4, 6...)

The overall product is positive.

The count of negative factors is an odd number (1, 3, 5, 7...)

The overall product is negative.



Worked Numeric Examples and Common Mistakes

Careful attention must be given to distinguishing multiplication rules from addition rules, as well as recognizing the distinction between expressions with grouping parentheses and expressions without them.

- **Worked Example 1:** $(-48) \div (-6) = +8$ (Same sign quotient yields positive)
- **Worked Example 2:** $(-7) \times 9 = -63$ (Different signs yield negative)
- **Worked Example 3:** $(-2) \times (-5) \times (-3) = -30$ (Three negative factors yield negative)
- **Worked Example 4:** $(-4)^2 = (-4) \times (-4) = +16$ (The base includes the negative sign)
- **Worked Example 5:** $-4^2 = -(4 \times 4) = -16$ (The exponent applies strictly to 4 before negation)



Core Sign Rules for Multiplication and Division

Multiplying and dividing signed integers follows consistent mathematical laws based on whether the signs of the two numbers are identical or opposite. When combining two factors or a dividend and divisor with matching signs, the result is always positive. When combining two terms with differing signs, the resulting product or quotient is always negative.

Positive × Positive = Positive:	$6 \times 4 = 24$
Negative × Negative = Positive:	$(-7) \times (-6) = 42$
Positive × Negative = Negative	$8 \times (-3) = -24$
Negative × Positive = Negative:	$(-3) \times 5 = -15$
Same sign quotient:	$(-24) \div (-8) = 3$
Opposite sign quotient:	$(-30) \div 5 = -6$



Universal Two-Term Rule

For any two real numbers a and b :
 $(-a) \times (-b) = ab$, and $(-a) \times b = -(ab)$.
 Similarly, $(-a) \div (-b) = \frac{a}{b}$, and $(-a) \div b = -(a/b)$ where $b \neq 0$.



Step-by-Step Approach

1

Step 1: Multiply or Divide Absolute Values

Ignore the positive and negative signs temporarily. Calculate the basic arithmetic product or quotient using the absolute values of the numbers.

2

Step 2: Inspect the Signs

Count the number of negative signs present across the pair of numbers being multiplied or divided.

3

Step 3: Assign the Final Sign

If both terms share the same sign (both positive or both negative), the result is positive. If the terms have opposite signs, append a negative sign to the result.



Common Mistake

Do not confuse sign rules for multiplication with addition. $(-3) + (-5) = -8$ (adding debts), but $(-3) \times (-5) = +15$ (two negatives multiplied yield a positive).

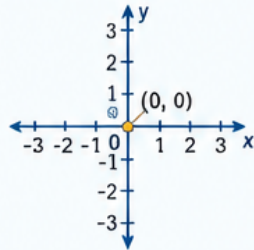
Mind Map: The Coordinate Plane, Quadrants, and Ordered Pairs



The Cartesian Coordinate Plane Anatomy

The coordinate plane is a two-dimensional grid formed by two perpendicular intersecting number lines. The horizontal axis is the x-axis, and the vertical axis is the y-axis. Their point of intersection is called the origin, written as the ordered pair $(0, 0)$. Every position on the plane corresponds to an ordered pair (x, y) that specifies its horizontal and vertical displacement from the origin.

- **x-axis:** The horizontal number line running left to right through 0.
- **y-axis:** The vertical number line running bottom to top through 0.
- **Origin:** The central reference point located at coordinates $(0, 0)$.
- **Scale:** Uniform intervals marked on each axis representing integers.



Orientation Check

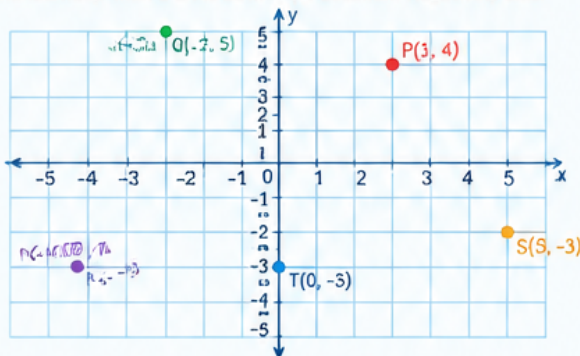
Always read or plot the horizontal value first along the x-axis, followed by the vertical value along the y-axis.



Worked Coordinate Mapping Examples

Review these concrete numeric examples demonstrating movement and quadrant placement across all four quadrants and axes.

- **Example A:** Point $P(3, 4)$ — Start at $(0, 0)$, move 3 units right $(+x)$, then 4 units up $(+y)$. Location: Quadrant I.
- **Example B:** Point $Q(-2, 5)$ — Start at $(0, 0)$, move 2 units left $(-x)$, then 5 units up $(+y)$. Location: Quadrant II.
- **Example C:** Point $R(-4, -3)$ — Start at $(0, 0)$, move 4 units left $(-x)$, then 3 units down $(-y)$. Location: Quadrant III.
- **Example D:** Point $S(5, -2)$ — Start at $(0, 0)$, move 5 units right $(+x)$, then 2 units down $(-y)$. Location: Quadrant IV.
- **Example E:** Point $T(0, -3)$ — Start at $(0, 0)$, move 0 units horizontally, then 3 units down $(-y)$. Location: Negative y-axis.



Common Mistake: Transposing Coordinates (y, x)

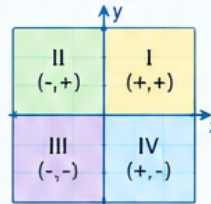
A frequent error is reversing coordinates, such as plotting $(2, 4)$ by moving up 2 and right 4. Remember alphabetical order: x comes before y, so horizontal movement always precedes vertical movement.



Four Quadrants and Sign Conventions

The intersecting axes divide the coordinate plane into four regions called quadrants. Quadrants are traditionally numbered counterclockwise using Roman numerals (I, II, III, IV), beginning at the upper right where both coordinates are strictly positive.

- **Quadrant I (Top Right):** Both coordinates positive $(+x, +y)$.
- **Quadrant II (Top Left):** Negative x and positive y $(-x, +y)$.
- **Quadrant III (Bottom Left):** Both coordinates negative $(-x, -y)$.
- **Quadrant IV (Bottom Right):** Positive x and negative y $(+x, -y)$.
- **Axis Locations:** Points with $y = 0$ lie on the x-axis; points with $x = 0$ lie on the y-axis. Points on axes do not belong to any quadrant.



Counterclockwise Order

Quadrant numbers follow a "C" shape: start top-right (I), move left to top-left (II), drop to bottom-left (III), and move right to bottom-right (IV).



Given an ordered pair (x, y) , where does it lie?

- $x > 0$ and $y > 0$ → Plot the point in Quadrant I.
- $x < 0$ and $y > 0$ → Plot the point in Quadrant II.
- $x < 0$ and $y < 0$ → Plot the point in Quadrant III.
- $x > 0$ and $y < 0$ → Plot the point in Quadrant IV.
- $x = 0$ or $y = 0$ → Plot the point directly on an axis (or at the origin if both are 0).



Step-by-Step Procedure for Plotting Ordered Pairs

An ordered pair is always written in parentheses separated by a comma: (x, y) . Plotting requires moving sequentially from the origin according to the signed values of x and y.

1

Start at the Origin

Place your pencil point at the origin $(0, 0)$ before making any directional movement.

2

Move Horizontally for x

Look at the first number (x). If positive, move that many units to the right. If negative, move that many units to the left. If zero, stay at the origin horizontally.

3

Move Vertically for y

From your new horizontal position, look at the second number (y). If positive, move up that many units. If negative, move down that many units. If zero, remain on the x-axis.

4

Mark and Label the Coordinate

Draw a clear dot at the final location and write the ordered pair (x, y) beside it.



Student Coordinate Practice Log

Practice identifying quadrants and plotting points with standard sign conventions. Record the quadrant or axis for each given point.

Point A $(-5, 2)$: Quadrant or Axis _____

Point B $(4, -6)$: Quadrant or Axis _____

Point C $(0, 5)$: Quadrant or Axis _____

Point D $(-3, -1)$: Quadrant or Axis _____



Algebraic Expressions, Variables, and Combining Like Terms



Algebraic Anatomy and Core Terminology

An algebraic expression contains constants, variables, coefficients, operations, and terms. It does not contain an equals sign.

- **Variable:** A letter (such as x , y , or n) representing an unknown value or quantity that can vary.
- **Constant:** A fixed numerical value that does not change (for example, 7 or -12).
- **Coefficient:** The numerical factor multiplying a variable (in $5x$, 5 is the coefficient; if no number is shown, such as x , the coefficient is 1).
- **Term:** A single number, a single variable, or numbers and variables multiplied together, separated from other terms by addition or subtraction signs.
- **Exponent:** The power indicating how many times a variable is multiplied by itself (in x^3 , 3 is the exponent).



Expression vs. Equation

An expression like $3x + 7$ represents a mathematical quantity without an equality relation. An equation like $3x + 7 = 22$ asserts that two expressions are equal and can be solved.



Identifying and Grouping Like Terms

Like terms are terms that have the exact same variables raised to the exact same exponents. Numerical constants are also like terms with each other.

- $3x$ and $7x$ are like terms (same variable x to power 1).
- $4x^2$ and $-9x^2$ are like terms (same variable x to power 2).
- $5x$ and $5y$ are **UNLIKE** terms (different variables).
- $6x^2$ and $6x$ are **UNLIKE** terms (different exponents).



Common Mistake Alert

Never combine terms with different variable powers into a single term. For example, $4x^2 + 3x$ does NOT equal $7x^3$ or $7x^2$. It cannot be simplified further.

1

Inspect and Identify

Scan the entire expression and identify terms that share identical variables and powers.

2

Group Along with Signs

Rearrange terms placing like terms adjacent to one another, keeping each sign (+ or -) attached to the term that immediately follows it.

3

Combine Coefficients

Add or subtract the coefficients of each like group while keeping the variable and exponent unchanged.

4

Write in Standard Form

List terms in descending order of exponents, placing constant numbers at the end.

Algebraic Expressions



Worked Numeric Example: Simplifying an Expression

Step-by-step reduction of the algebraic expression:

Start $5x^2 + 8x - 2x^2 + 3 + 4x - 9$

- 1 **Original Expression:**
 $5x^2 + 8x - 2x^2 + 3 + 4x - 9$
- 2 **Group x^2 terms:**
 $(5x^2 - 2x^2) = 3x^2$
- 3 **Group x terms:**
 $(+8x + 4x) = +12x$
- 4 **Group constant numbers:**
 $(+3 - 9) = -6$
- 5 **Final Simplified Expression:**
 $3x^2 + 12x - 6$



Sign Attachment Tip

Always treat a minus sign as a negative coefficient when grouping. For example, in $4x - 9$, the constant term is -9.



Quick Review Practice & Check

Apply the grouping steps to verify whether pairs of terms can be combined.

- Can $8a$ and $-14a$ be combined? Yes, they have the identical variable a .
- Can $7x^2$ and $7y^2$ be combined? No, they have different base variables x and y .
- Can $12m$ and $5m^2$ be combined? No, their exponents (1 and 2) do not match.



Try It!

Simplify: $7y + 4x - 2y + 6 - x$

Simplify: $3a^2 + 5a - a^2 + 8$



Common Mistake

Do two terms share both the exact same variable letter and the exact same power exponent?



Yes (e.g., $6x^2$ and $-2x^2$)

Add or subtract their numeric coefficients and keep the variable base and exponent identical.



No (e.g., $6x^2$ and $3x$, or $4x$ and $4y$)

Keep terms separated by their respective operation signs; do not combine their coefficients.



One-Step Linear Equations Mind Map



Core Principle: The Balance Scale Model

An algebraic equation represents a balanced scale where the left-hand expression equals the right-hand expression. To maintain equality, any arithmetic operation applied to one side must be applied identically to the other side.

- **Equality condition:** If $A = B$, then $A + c = B + c$ and $A - c = B - c$.
- **Uniform scaling:** If $A = B$, then $A * c = B * c$ and $A / c = B / c$ (for $c \neq 0$).
- **Goal of isolation:** Manipulate the equation until the unknown variable stands alone with a coefficient of 1.



Golden Rule of Equations

Whatever operation is performed on one side of the equals sign must be performed identically on the other side to keep the relationship true.

Steps to Solve One-Step Equations

1. **Identify Variable and Operation**
Locate the unknown letter and observe which arithmetic operation connects it to an adjacent numeric constant.
2. **Select the Inverse Operation**
Choose the exact mathematical inverse: addition reverses subtraction, and multiplication reverses division.
3. **Apply to Both Sides**
Execute the inverse operation on both the left-hand side and right-hand side to isolate the variable.
4. **Perform Algebraic Verification**
Substitute the calculated value back into the original equation to verify that both sides evaluate to identical quantities.



Additive and Subtractive Inverses

Addition and subtraction are inverse operations.

An additive inverse neutralizes a constant term to zero via the identity property ($a + (-a) = 0$).

Examples

- **Solving Addition:** $x + 9 = 21 \rightarrow$ Subtract 9 from both sides $\rightarrow x = 12$.
- **Verification:** $12 + 9 = 21$ (True).
- **Solving Subtraction:** $y - 14 = 32 \rightarrow$ Add 14 to both sides $\rightarrow y = 46$.
- **Verification:** $46 - 14 = 32$ (True).



Which operation is applied to the variable term?

A constant is added to the variable
(e.g., $x + b = c$)

A constant is subtracted from the variable
(e.g., $x - b = c$)

Subtract b from both sides of the equation.

Add b to both sides of the equation.

One-Step Linear Equations



Multiplicative and Divisive Inverses

Multiplication and division are inverse operations. A multiplicative inverse scales a non-zero coefficient to 1 via the property $a * (1/a) = 1$.

Examples

- **Solving Multiplication:** $6m = 48 \rightarrow$ Divide both sides by 6 $\rightarrow m = 8$.
- **Verification:** $6 * 8 = 48$ (True).
- **Solving Division:** $k/5 = 7 \rightarrow$ Multiply both sides by 5 $\rightarrow k = 35$.
- **Verification:** $35/5 = 7$ (True).



How is the variable linked to its numeric partner?

The variable has a multiplying coefficient (e.g., $a * x = b$)

The variable is being divided (e.g., $x / a = b$)

Divide both sides by the non-zero coefficient a .

Multiply both sides by the non-zero divisor a .



Common Mistakes: Unilateral Operations

A unilateral operation occurs when a student applies an arithmetic change to only one side of the equals sign or applies mismatched operations. This violates algebraic equality.

Examples

- **Unilateral Error:** Given $x + 7 = 15$, subtracting 7 from only the left gives $x = 15$ (False, $15 + 7 \neq 15$).
- **Sign Error:** Adding instead of subtracting, producing $x = 22$ when the equation requires inverse neutralization.
- **Check Routine:** Always substitute the answer into the original expression to catch arithmetic or sign errors immediately.



Warning: Unilateral Modification

Never alter one side of an equation without executing the identical arithmetic computation on the opposite side.

Practice: Fill in the Blanks

Original Equation Check Expression:

Left-Hand Side Substitution Value:

Right-Hand Side Target Equality:

Two-Step Linear Equations and Multi-Step Simplification



Reverse Order of Operations (SADMEP)

To isolate an unknown variable in a standard two-step linear equation, reverse the standard order of operations (PEMDAS/BODMAS). First undo any addition or subtraction attached to the variable term, then undo any multiplication or division.

- 1 **Step 1:** Identify the variable term and isolate it by applying the inverse operation of addition or subtraction to both sides.
- 2 **Step 2:** Balance the equation by performing the exact same arithmetic operation on the opposite side.
- 3 **Step 3:** Eliminate the coefficient by multiplying or dividing both sides by that exact value.
- 4 **Step 4:** Check your solution by substituting the calculated value back into the original algebraic expression.



Common Mistake Alert

Do not divide before clearing additions or subtractions unless the entire side is enclosed in parentheses or divided by a single denominator. Dividing first often leads to fraction calculation errors.

1 Identify Operations

In the equation $3x + 7 = 22$, note that the variable x is multiplied by 3 and then added to 7.

2 Undo Addition

Subtract 7 from both sides of the equation:
 $3x + 7 - 7 = 22 - 7$,
 simplifying directly to
 $3x = 15$.

3 Undo Multiplication

Divide both sides by the coefficient 3:
 $\frac{3x}{3} = \frac{15}{3}$
 yielding the single variable solution $x = 5$.

4 Verify by Substitution

Substitute $x = 5$ back into the original equation:
 $3(5) + 7 = 15 + 7 = 22$.
 Both sides match, confirming the result.



Multi-Step Simplification and the Distributive Property

When linear equations contain parentheses, fractional bars, or multiple variable terms, simplify each side of the equation completely before isolating the unknown variable.

- Distribute coefficients across parentheses: $a(b + c) = ab + ac$, preserving negative signs accurately.
- Combine like terms on the same side of the equal sign before moving terms across sides.
- When dividing by a negative coefficient in the final step, ensure the sign of the quotient is flipped correctly.

! Negative Sign Error

Remember that distributing a negative number changes the sign of every term inside the parentheses: $-2(x - 4)$ becomes $-2x + 8$, not $-2x - 8$.

Two-Step & Multi-Step Equations



Distributive Property & Like Terms

1 Distribute Grouping Symbols

Given $2(4x - 3) + 5 = 23$, distribute the 2 across the binomial:
 $8x - 6 + 5 = 23$.

2 Combine Like Terms

Combine the constant integers on the left-hand side: $-6 + 5 = -1$, rewriting the simplified equation as $8x - 1 = 23$.

3 Isolate the Variable Term

Add 1 to both sides: $8x - 1 + 1 = 23 + 1$, which reduces the linear relationship to $8x = 24$.

4 Solve and Check

Divide both sides by 8: $x = 3$. Check:
 $2(4(3) - 3) + 5 = 2(12 - 3) + 5 = 2(9) + 5 = 18 + 5 = 23$.



Does the equation have parentheses or like terms on one side before isolating?

Yes, grouping symbols or like terms exist

Apply the distributive property to eliminate parentheses and combine like terms before undoing addition and subtraction.

No, it is in standard $ax + b = c$ form

Proceed directly to inverse operations: undo addition or subtraction first, followed by multiplication or division.



Practice: Solve $-4(2x + 5) = 12$
 (Show all inverse steps)

- 1 _____
- 2 _____
- 3 _____
- 4 _____

One-Step and Two-Step Inequalities & Number Line Graphing



Inequality Symbols & Boundary Conditions

Inequalities compare two algebraic expressions using relative order relations. Boundary values are plotted on a 1D real number line using specific endpoint circles to indicate whether the endpoint itself is included in the solution set.

- **Strict Less Than ($<$):** Values strictly to the left; plotted with an open circle (O) at the boundary value.
- **Strict Greater Than ($>$):** Values strictly to the right; plotted with an open circle (O) at the boundary value.
- **Less Than or Equal To ($\leq$):** Values to the left including boundary; plotted with a solid closed circle (●).
- **Greater Than or Equal To ($\geq$):** Values to the right including boundary; plotted with a solid closed circle (●).



Endpoint Convention

An open circle indicates strict inequality (endpoint excluded). A solid closed circle indicates non-strict inequality (endpoint included).



Solving Linear Inequalities Step-by-Step

Solving linear inequalities follows standard inverse-operation algebra with one crucial operational rule: multiplying or dividing both sides by a negative number reverses the inequality relation.



Mandatory Sign Reversal Rule

Whenever multiplying or dividing both sides of an inequality by a negative quantity, immediately invert the direction of the inequality symbol (e.g., $<$ becomes $>$, and $\geq$ becomes $\leq$).

1

Isolate the Variable Term

Use addition or subtraction properties of inequality to eliminate constant terms from the variable side.

2

Eliminate the Coefficient

Divide or multiply both sides by the numerical coefficient. If that coefficient is negative, reverse the direction of the inequality symbol.

3

Graph the Solution Set

Locate the boundary value on the real number line, place an open or closed circle according to the relation type, and draw a shaded ray extending in the solution direction.

4

Verify with a Test Value

Select an integer from the shaded region, substitute it into the original inequality, and verify that the resulting statement evaluates to true.



Worked Numeric Examples

Complete algebraic solutions and verified number line representations for standard one-step and two-step inequalities.

1. One-Step Example with Negative Division

Solve $-4x < 12$. Divide both sides by -4 and reverse symbol: $x > -3$.

Number line representation: open circle at -3 , ray shaded to the right towards $+\infty$.



2. Two-Step Standard Example

Solve $2x + 5 \geq 13$. Subtract 5 from both sides to get $2x \geq 8$. Divide both sides by positive 2: $x \geq 4$.

Number line representation: solid closed circle at 4, ray shaded to the right towards $+\infty$.



3. Two-Step Negative Coefficient Example

Solve $-3x + 7 > 22$. Subtract 7 from both sides: $-3x > 15$. Divide both sides by -3 and reverse the inequality sign: $x < -5$.

Number line representation: open circle at -5 , ray shaded to the left towards $-\infty$.



Is the operation multiplying or dividing both sides by a negative value?

Yes, multiplying or dividing by a negative number

Invert the inequality symbol immediately ($<$ to $>$, $>$ to $<$, $\leq$ to $\geq$, $\geq$ to $\leq$).

No, adding/subtracting any number, or multiplying/dividing by a positive number

Preserve the current inequality symbol direction unchanged.

Inequalities & Graphing



Practice & Verification Checklist

Complete these verification checks when solving and graphing linear inequalities.

☐ Boundary point value and circle type (Open / Closed)

☐ Ray direction arrow (Left for $<$ / $\leq$, Right for $>$ / $\geq$)

☐ Selected test point and verification check result



Common Mistake Alert

Do not invert the inequality symbol merely because a negative constant appears on the opposite side; reversal **ONLY** occurs when multiplying or dividing by a negative value.

Geometric Foundations: Angles, Parallel Lines, and Transversals



Angle Fundamentals & Core Classifications

An angle measures the amount of turn between two intersecting rays sharing a common endpoint called the vertex. Standard geometric classification uses degree measures from 0° up to 180° for planar lines.

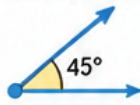
- **Acute Angle:** Greater than 0° and strictly less than 90° .
- **Right Angle:** Exactly 90° , forming perpendicular line segments denoted by a square corner symbol.
- **Obtuse Angle:** Strictly greater than 90° and strictly less than 180° .
- **Straight Angle:** Exactly 180° , forming a single collinear straight line.



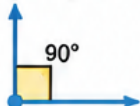
Vertex Labeling Rule

When naming an angle with three letters such as $\angle ABC$, the middle letter B must always denote the vertex where the two rays meet.

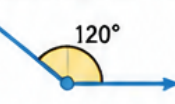
Acute Angle ($< 90^\circ$)



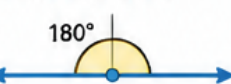
Right Angle (90°)



Obtuse Angle ($90^\circ < \theta < 180^\circ$)



Straight Angle (180°)



Angle Pairs: Complementary, Supplementary, and Vertical

Relationships between pairs of angles allow unknown values to be determined without direct measurement using protractors.

- **Complementary Angles:** Two angle measures whose sum equals exactly 90° (e.g., $35^\circ + 55^\circ = 90^\circ$).
- **Supplementary Angles:** Two angle measures whose sum equals exactly 180° (e.g., $110^\circ + 70^\circ = 180^\circ$).
- **Linear Pair:** Adjacent supplementary angles resting on a single straight line whose sum is 180° .
- **Vertical Angles:** Opposite pairs formed by two intersecting lines; vertical angles are always congruent (equal in measure).



Common Mistake

Do not assume complementary and supplementary angles must share a common side or vertex; their classification depends solely on whether their degree sum equals 90° or 180° .

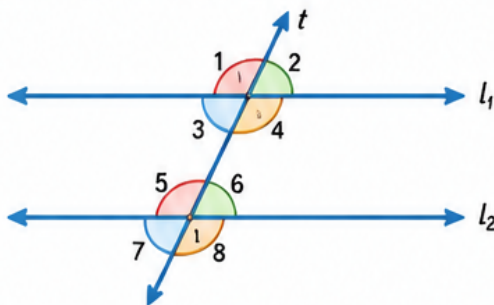
Geometric Foundations: Angles & Lines



Parallel Lines Cut by a Transversal

When two coplanar parallel lines ($l_1 \parallel l_2$) are intersected by a third line called a transversal (t), eight distinct angles are formed into predictable congruent and supplementary groupings.

- **Corresponding Angles:** Occupy the same relative position at each intersection; they are congruent (e.g., top-right equals top-right).
- **Alternate Interior Angles:** Lie on opposite sides of the transversal inside the two parallel lines; they are congruent.
- **Alternate Exterior Angles:** Lie on opposite sides of the transversal outside the parallel lines; they are congruent.
- **Consecutive Interior Angles:** Lie on the same side of the transversal between the parallel lines; they are supplementary (sum = 180°).



Parallelism Requirement

These angle equivalencies hold true strictly when the lines being cut are parallel. If the lines are not parallel, these named geometric positions still exist, but their angle measures will not be congruent or supplementary.



Worked Numeric Example: Solving for Unknown Angles

Given two parallel lines cut by transversal line t , where an exterior angle measures $(3x + 15)^\circ$ and its alternate exterior angle measures $(5x - 25)^\circ$. Find x and all corresponding angle measures.

- **Step 1:** Equate alternate exterior angles: $3x + 15 = 5x - 25$.
- **Step 2:** Subtract $3x$ from both sides: $15 = 2x - 25$.
- **Step 3:** Add 25 to both sides: $40 = 2x$, so $x = 20$.
- **Step 4:** Substitute x : $3(20) + 15 = 75^\circ$. The alternate exterior angles both measure 75° .
- **Step 5:** Determine adjacent supplementary angle: $180^\circ - 75^\circ = 105^\circ$.



Verification Check

Check: $5(20) - 25 = 100 - 25 = 75^\circ$.

The two angles match, confirming the solution $x = 20$ is accurate.

Steps to Solve

- 1 **Identify Geometric Relationship**
Observe whether the given angle positions are congruent (alternate interior, alternate exterior, corresponding, vertical) or supplementary (linear pair, consecutive interior).
- 2 **Set Up Algebraic Equation**
Form an equality $\text{Angle A} = \text{Angle B}$ if congruent, or $\text{Angle A} + \text{Angle B} = 180^\circ$ if supplementary.
- 3 **Solve for Variable and Verify**
Isolate x using standard inverse operations, substitute back into both angle expressions, and ensure adjacent angles sum to 180° .

Triangles: Classification, Interior Angle Sum, and the Pythagorean Theorem



Triangles Overview & Radial Map Structure

A triangle is a closed three-sided two-dimensional polygon defined by three line segments meeting at three vertices. Every triangle can be classified concurrently along two independent dimensions: the relative lengths of its sides and the interior measures of its three angles. Additionally, Euclidean planar geometry guarantees that the sum of the interior angles of any planar triangle is exactly 180 degrees. For right-angled triangles, the geometric relationship between the two perpendicular legs and the opposite hypotenuse is strictly governed by the Pythagorean theorem.

- Classification by Side Lengths: Equilateral, Isosceles, and Scalene.
- Classification by Angle Measures: Acute, Right, and Obtuse.
- Angle Sum Property: Sum of interior angles $x^\circ + y^\circ + z^\circ = 180^\circ$.
- Pythagorean Theorem: In right triangles with perpendicular legs a and b and hypotenuse c , $a^2 + b^2 = c^2$.



Core Theorem

The interior angles of any triangle always add up to 180 degrees ($x^\circ + y^\circ + z^\circ = 180^\circ$). This relationship holds regardless of triangle size, shape, or orientation.



Classification by Sides and Angles

Every triangle possesses distinct structural attributes based on side lengths and angle measures. Classifying a triangle requires examining both properties simultaneously.

By Side Lengths



Equilateral Triangle

- All three sides are equal in length.
- All three angles are 60° .



Isosceles Triangle

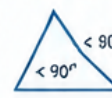
- At least two sides have equal length.
- Angles opposite equal sides are congruent.



Scalene Triangle

- All three sides have different lengths.
- All three angles are distinct.

By Angle Measures



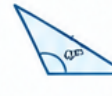
Acute Triangle

- All three interior angles are strictly less than 90° .



Right Triangle

- Exactly one angle is 90° .
- Opposite side is the hypotenuse.



Obtuse Triangle

- Exactly one angle is greater than 90° (and less than 180°).



Side Length Verification

Under the Triangle Inequality Theorem, the sum of the lengths of any two sides of a triangle must always be strictly greater than the length of the remaining side.



Calculating Unknown Interior Angles

When two interior angles of a triangle are known, the measure of the third angle can be solved directly using algebraic substitution based on the 180° angle sum property.

- Standard Formula: Angle A + Angle B + Angle C = 180° .
- Isolating Missing Angle: Angle C = $180^\circ - (\text{Angle A} + \text{Angle B})$.
- Special Case Right Triangle: Angle A + Angle B = 90° .

Worked Numeric Example: Interior Angles

Given a triangle with known angles measuring 48° and 67° , find the third angle x :

Step 1: Add known angles: $48^\circ + 67^\circ = 115^\circ$.

Step 2: Subtract from 180° : $x = 180^\circ - 115^\circ = 65^\circ$.

Verification: $48^\circ + 67^\circ + 65^\circ = 180^\circ$.

Steps:

- 1 Identify Known Angles**
Measure or extract the two given interior angles from the problem statement or diagram.
- 2 Sum the Known Values**
Add the two known angles together to find their combined measure.
- 3 Subtract from 180 Degrees**
Subtract the combined sum from 180° to calculate the exact measure of the missing angle.
- 4 Verify Total**
Sum all three resulting angles to ensure the final total equals exactly 180° .

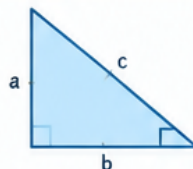
TRIANGLES: Types, Angles & Pythagorean Theorem



The Pythagorean Theorem & Hypotenuse Calculations

In any right-angled triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs. The hypotenuse is always the longest side and sits directly opposite the 90° right angle.

- Fundamental Formula: $a^2 + b^2 = c^2$, where a and b are legs, and c is the hypotenuse.
- Finding Hypotenuse: $c = \sqrt{a^2 + b^2}$.
- Finding Leg Length: $a = \sqrt{c^2 - b^2}$ or $b = \sqrt{c^2 - a^2}$.



$a, b = \text{legs}$
 $c = \text{hypotenuse}$



Common Calculation Trap

Never add leg lengths before squaring!

$$\sqrt{a^2 + b^2} \neq a + b.$$

For example, $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$, whereas $3 + 4 = 7$.



Problem Solving Verification Checklist

Use this structured verification check to categorize any given triangle and verify that side lengths or angle calculations are algebraically and geometrically consistent.

- Confirm all angle values are non-negative and sum to exactly 180° .
- Ensure the hypotenuse in a right triangle is strictly longer than each individual leg.
- Check that acute angles in a right triangle sum to 90° .

Given Side Lengths (a, b, c):

Side Classification (Equilateral / Isosceles / Scalene):

Interior Angles ($x^\circ, y^\circ, z^\circ$) & Angle Classification:

Pythagorean Equation Check (if right triangle: $a^2 + b^2 = c^2$):

Decision

Does the triangle contain a confirmed 90° right angle?

Yes, exactly one angle is 90° → Classify as a right triangle. The remaining two angles must sum to 90° . Apply the Pythagorean theorem ($a^2 + b^2 = c^2$) to relate the legs to the hypotenuse.

No, all angles are strictly less than 90° → Classify as an acute triangle. Confirm interior sum equals 180° . The Pythagorean equality will satisfy $a^2 + b^2 < c^2$ where c is the longest side.

No, one angle is greater than 90° → Classify as an obtuse triangle. Confirm interior sum equals 180° . The relationship will satisfy $a^2 + b^2 > c^2$ where c is the longest side.

Perimeter and Area of Polygons: Rectangles, Triangles, and Parallelograms



Boundary vs. Surface Area

Perimeter measures the total distance around the outer boundary of a two-dimensional closed figure, expressed in linear units (such as cm, m, or inches). Area measures the amount of interior two-dimensional surface space enclosed within that boundary, always expressed in square units (such as cm^2 , m^2 , or sq in).

- **Linear Boundary (P):** Calculated by finding the sum of all exterior side lengths.
- **Interior Area (A):** Calculated by tiling or multiplying perpendicular dimensions in square units.
- **Dimensional Consistency:** Never add linear units directly to square units; verify matching measurement scales before multiplying.



Unit Rule

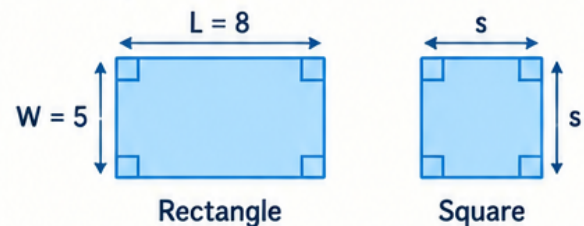
Perimeter uses linear measures (e.g., cm). Area represents 2D region coverage and must always be recorded in square units (e.g., cm^2).



Standard Polygon Formulas: Squares and Rectangles

Squares and rectangles have four straight boundary sides meeting at 90-degree right angles. Opposite sides of a rectangle are equal in length, while all four sides of a square are congruent.

- **Square Perimeter:** $P = 4s$ (where s is side length)
- **Square Area:** $A = s^2 = s \times s$
- **Rectangle Perimeter:** $P = 2L + 2W$ (where L is length and W is width)
- **Rectangle Area:** $A = L \times W$



Polygons: Perimeter & Area



Worked Numeric Examples

Verify calculations using established arithmetic substitution procedures across standard polygons.

1 Step 1: Rectangle Dimensions

Given length $L = 8$ cm and width $W = 5$ cm.

Perimeter: $P = 2(8 \text{ cm}) + 2(5 \text{ cm}) = 16 \text{ cm} + 10 \text{ cm} = 26 \text{ cm}$.

Area: $A = 8 \text{ cm} \times 5 \text{ cm} = 40 \text{ cm}^2$.

2 Step 2: Parallelogram with Slant Edge Trap

Given base $b = 10$ m, slant side $s = 6$ m, and perpendicular altitude $h = 4$ m.

Perimeter: $P = 2(10 \text{ m}) + 2(6 \text{ m}) = 20 \text{ m} + 12 \text{ m} = 32 \text{ m}$.

Area: $A = b \times h = 10 \text{ m} \times 4 \text{ m} = 40 \text{ m}^2$
(slant length 6 m is used for perimeter, not area).

3 Step 3: Triangle Calculations

Given a triangle with base $b = 12$ cm, side lengths $a = 10$ cm, $c = 10$ cm, and central altitude $h = 8$ cm.

Perimeter: $P = 10 \text{ cm} + 12 \text{ cm} + 10 \text{ cm} = 32 \text{ cm}$.

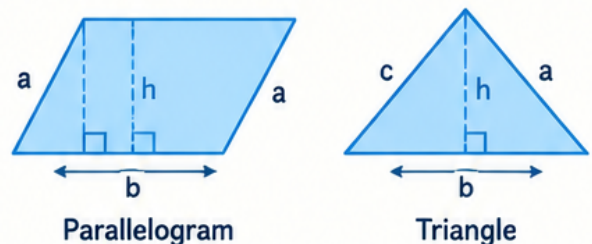
Area: $A = (1/2) \times b \times h = 0.5 \times 12 \text{ cm} \times 8 \text{ cm} = 48 \text{ cm}^2$.



Parallelograms & Triangles

Parallelograms and triangles require measuring base (b) and vertical perpendicular height (h). The perpendicular height is the altitude forming a 90° right angle to the chosen base line, never the slanted edge.

- **Parallelogram Perimeter:** $P = 2a + 2b$
(sum of all 4 outer boundary edges)
- **Parallelogram Area:** $A = b \times h$
(base multiplied by perpendicular height)
- **Triangle Perimeter:** $P = a + b + c$
(sum of the three outer sides)
- **Triangle Area:** $A = (1/2) \times b \times h$
(one-half of the base multiplied by perpendicular height)



Common Mistake

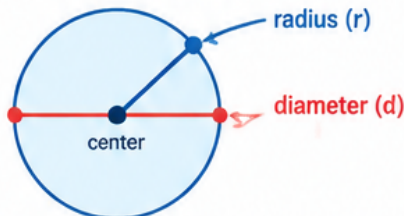
Do not multiply the base by the slant side length when calculating area.

Area formulas require the true perpendicular altitude (h) measured at 90 degrees to base (b).

Circles: Radius, Diameter, Circumference, and Area

1 Circle Anatomy & Linear Dimensions

A circle is defined by all points in a plane equidistant from a fixed center point. Its key linear measurements describe distances across, within, and around the boundary.



- **Radius (r):** The straight-line distance from the exact center to any point on the boundary.
- **Diameter (d):** A straight line passing directly through the center connecting two boundary points. Formula: $d = 2r$ (or $r = d / 2$).
- **Circumference (C):** The total perimeter or linear distance around the outside edge of the circle.
- **Pi (π):** The ratio of circumference to diameter (C / d), universally approximated as 3.14 or $22/7$.



Quick Identification Rule

If a line segment stops at the center, it is a radius. If it passes completely from edge to edge through the center, it is a diameter.

2 Calculating Circumference and Area

Formulas for perimeter and enclosed surface area depend entirely on the radius or diameter, scaled by the geometric constant π .

Circumference Formulas:

$$C = 2\pi r \text{ or } C = \pi d$$

Area Formula:

$$A = \pi r^2$$

(always expressed in square units).



- **Converting Diameter for Area:** Always compute $r = d / 2$ prior to evaluating the area equation.



Unit Awareness

Circumference is a one-dimensional linear measure (e.g., cm, in, m), whereas Area represents two-dimensional surface coverage and must be written in square units (e.g., cm^2 , sq in, m^2).

Circles: Anatomy & Formulas

3 Worked Numeric Example: Complete Circle Analysis

Step-by-step calculation for a circle with a given diameter of 14 cm, applying both circumference and area equations with $\pi \approx 22/7$ and $\pi \approx 3.14$.

1 Step 1: Identify Given Data and Find Radius

Given diameter $d = 14$ cm. Find radius r :

$$r = \frac{d}{2} = \frac{14}{2} = 7 \text{ cm.}$$

2 Step 2: Calculate Circumference (C)

Apply $C = 2\pi r$. Using $\pi \approx 22/7$:

$$C = 2 * \frac{22}{7} * 7 = 44 \text{ cm.}$$

(Using $\pi \approx 3.14$: $C \approx 3.14 * 14 = 43.96$ cm).

3 Step 3: Calculate Area (A)

Apply $A = \pi r^2$. Square the radius first:

$$r^2 = 7^2 = 49 \text{ cm}^2.$$

Multiply by π : $A = \frac{22}{7} * 49 = 22 * 7 = 154 \text{ cm}^2$.
(or $3.14 * 49 \approx 153.86 \text{ cm}^2$).

4 Common Mistakes & Pitfall Checklist

Watch for these frequent operational errors when solving circle problems in geometry.

- ⊗ **Squaring the Diameter:** Using πd^2 instead of πr^2 . Squaring the diameter yields an answer four times larger than the correct area.
- ⊗ **Doubling Instead of Squaring:** Multiplying the radius by 2 ($2r$) instead of squaring it ($r * r$) when evaluating area. Note: $2\pi r$ is circumference, not area.
- ⊗ **Missing Square Units:** Labeling area as linear units (cm) instead of square units (cm^2).
- ⊗ **Formula Confusion:** Forgetting that $C = 2\pi r$ contains a factor of 2, while $A = \pi r^2$ contains an exponent of 2.



Red Alert: The '2r vs r²' Trap

Radius squared (r^2) means r multiplied by itself ($r * r$). Never multiply r by 2 when computing area! Only when $r = 2$ do r^2 and $2r$ equal the same numerical value (4).

Mind Map: Surface Area and Volume (Rectangular Prisms & Cylinders)



3D Spatial Measurement: Core Foundations

Three-dimensional figures occupy physical space defined by three distinct dimensions: length, width, and height. Measuring 3D solids involves two fundamentally distinct metrics: **surface area** (the total two-dimensional boundary covering the outside of the solid) and **volume** (the internal space or three-dimensional capacity enclosed within the solid).

- **Dimensions:** 1D measures length (units), 2D measures boundary surface (square units), and 3D measures capacity (cubic units).
- **Rectangular Prism:** A polyhedron with 6 rectangular faces meeting at right angles, featuring 12 edges and 8 vertices.
- **Cylinder:** A curved geometric solid with two parallel, congruent circular bases connected by a curved lateral surface.



Rectangular prism
(3D)



Cylinder
(3D)



Dimensional Unit Distinction

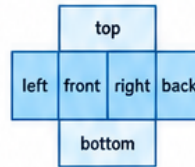
Never confuse square units with cubic units. Surface area represents outer wrapping and is recorded as square units (such as cm^2 or in^2). Volume represents interior capacity and is recorded as cubic units (such as cm^3 or in^3).



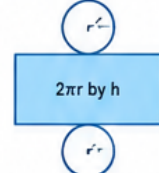
Surface Area: Net Unfolding & Formulas

Surface area (SA) is calculated by unfolding the 3D solid into a flat two-dimensional net, finding the area of each distinct polygon or circular face, and summing them together.

- **Rectangular Prism Net:** Unfolds into 3 pairs of congruent rectangles (top/bottom, front/back, left/right).
Total SA = $2(l \cdot w) + 2(l \cdot h) + 2(w \cdot h)$.
- **Cylinder Net:** Unfolds into two congruent circles of radius r and one unrolled rectangular lateral face with length equal to the circular circumference ($2\pi r$) and height h .
- **Cylinder Lateral Area:** $A_L = 2 \cdot \pi \cdot r \cdot h$
- **Cylinder Total Surface Area:** $SA = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h$



Rectangular Prism Net
(6 faces)



Cylinder Net
(2 circles + 1 rectangle)



Common Mistake: Cylinder Net Width

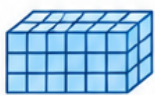
A frequent error is using diameter or radius instead of base circumference for the width of the unrolled cylinder rectangle. The unrolled curved wall always stretches to exactly $2 \cdot \pi \cdot r$.



Volume: 3D Enclosed Capacity

Volume (V) measures the count of unit cubes required to fill the interior of a 3D solid without gaps or overlaps. For uniform prisms and cylinders, volume equals the base area (B) multiplied by the perpendicular height (h).

- **Rectangular Prism Base Area:** $B = l \cdot w$
- **Rectangular Prism Volume:** $V = B \cdot h = l \cdot w \cdot h$
- **Cylinder Circular Base Area:** $B = \pi \cdot r^2$
- **Cylinder Volume:** $V = B \cdot h = \pi \cdot r^2 \cdot h$



Unit cubes filling prism base
 $B = l \cdot w$



Unit cubes filling cylinder
base $B = \pi \cdot r^2$



Common Mistake: Exponent Placement

In cylinder volume ($V = \pi r^2 h$), only the radius is squared. Do not square the height, and do not multiply by π before squaring the radius.

Surface Area & Volume



Step-by-Step Numeric Walkthroughs

Systematic evaluation of surface area and volume for standard rectangular prisms and right cylinders using defined parameter values.

- Always calculate the area of the base first before extending calculations to height.
- Substitute $\pi \approx 3.14159$ or $22/7$ when a decimal approximation is requested, or keep π exact when required.

- 1 **Prism SA Calculation** ($l = 6 \text{ cm}$, $w = 4 \text{ cm}$, $h = 5 \text{ cm}$)
Evaluate face pairs: $2(6 \cdot 4) = 48 \text{ cm}^2$; $2(6 \cdot 5) = 60 \text{ cm}^2$;
 $2(4 \cdot 5) = 40 \text{ cm}^2$. Sum pairs: $SA = 48 + 60 + 40 = 148 \text{ cm}^2$.
- 2 **Prism Volume Calculation** ($l = 6 \text{ cm}$, $w = 4 \text{ cm}$, $h = 5 \text{ cm}$)
Calculate base area $B = 6 \cdot 4 = 24 \text{ cm}^2$.
Multiply by height: $V = 24 \cdot 5 = 120 \text{ cm}^3$.
- 3 **Cylinder SA Calculation** ($r = 3 \text{ cm}$, $h = 7 \text{ cm}$, $\pi \approx 3.14$)
Two circular bases: $2 \cdot \pi \cdot (3)^2 = 2 \cdot 3.14 \cdot 9 = 56.52 \text{ cm}^2$.
Lateral face: $2 \cdot \pi \cdot 3 \cdot 7 = 42 \cdot 3.14 = 131.88 \text{ cm}^2$.
Total SA = $56.52 + 131.88 = 188.4 \text{ cm}^2$.
- 4 **Cylinder Volume Calculation** ($r = 3 \text{ cm}$, $h = 7 \text{ cm}$, $\pi \approx 3.14$)
Calculate base area $B = \pi \cdot (3)^2 = 9 \cdot 3.14 = 28.26 \text{ cm}^2$.
Multiply by height: $V = 28.26 \cdot 7 = 197.82 \text{ cm}^3$.



Metric Decision Guide: Surface Area vs Volume

Choose the appropriate measurement metric based on real-world problem context.

- **Surface Area** applies to wrapping, painting, coating, and exterior coverage.
- **Volume** applies to liquid filling, packaging capacity, interior space, and material mass.

What physical property does the practical scenario ask you to measure?

Covering the outer faces of a box with wrapping paper or painting a cylindrical tank

Calculate **Surface Area** using square units: $SA = 2lw + 2lh + 2wh$ for prisms, or $SA = 2\pi r^2 + 2\pi rh$ for cylinders.

Determining how much water fills a cylindrical tank or how many unit blocks fit inside a carton

Calculate **Volume** using cubic units:
 $V = l \cdot w \cdot h$ for prisms,
or $V = \pi r^2 h$ for cylinders.



Independent Practice & Verification Log

Students or tutors record independent calculations to verify mastery of spatial formulas and units.

- Ensure measurements are expressed in identical units before executing multiplication.
- Verify final answers include explicit cm^2 , m^2 , cm^3 , or m^3 unit labels.

- 1 Prism Practice: Dimensions ($l = 8 \text{ cm}$, $w = 5 \text{ cm}$, $h = 10 \text{ cm}$)
→ Total SA (cm^2) _____
- 2 Prism Practice: Dimensions ($l = 8 \text{ cm}$, $w = 5 \text{ cm}$, $h = 10 \text{ cm}$)
→ Volume (cm^3) _____
- 3 Cylinder Practice: Dimensions ($r = 4 \text{ cm}$, $h = 6 \text{ cm}$, $\pi \approx 3.14$)
→ Total SA (cm^2) _____
- 4 Cylinder Practice: Dimensions ($r = 4 \text{ cm}$, $h = 6 \text{ cm}$, $\pi \approx 3.14$)
→ Volume (cm^3) _____
- 5 Unit Check: Did you record square units for SA and cubic units for Volume? _____



Measures of Central Tendency and Data Spread



Mean (Average)

The arithmetic mean combines every data point into an aggregate sum and distributes it evenly across the total number of entries.

$$\bar{x} = \frac{\sum x}{n}$$

Example:

For the data {56, 69, 82, 87, 87, 93}:

$$\sum x = 474, n = 6$$

$$\bar{x} = \frac{474}{6} = 79$$



Mode (Frequency)

The mode is the most frequently occurring observation or observations within the data.

Example:

From {56, 69, 82, 87, 87, 93}
87 appears twice (more than any other value).

$$\text{Mode} = 87$$



Median (Center Value)

The median is the physical center of an ordered list, dividing data into equal halves.

Ordered data: {56, 69, 82, 87, 87, 93}

{56, 69, 82, 87, 87, 93}

Middle pair

$$\text{Median} = \frac{82 + 87}{2} = \frac{169}{2} = 84.5$$



Range (Spread)

The range is the difference between the maximum and minimum values in the data set.

Example:

Maximum = 93, Minimum = 56

$$\text{Range} = 93 - 56 = 37$$



Central Tendency & Spread



Central Tendency & Spread Overview

Data summaries describe the center and the dispersion of a data set using distinct numerical algorithms. The mean, median, and mode identify typical values, while the range calculates the total distance between extreme values.

- **Mean:** Arithmetic average calculated by summing all observations and dividing by the total count.
- **Median:** Physical center of an ordered list, dividing data into equal halves.
- **Mode:** The most frequently occurring observation or observations within the data.
- **Range:** Difference between the maximum and minimum values in the data set.



Outlier Sensitivity

The mean and range are highly sensitive to extreme values (outliers), whereas the median remains stable and robust when skewed values are present.



1 Collect and Organize

Record all raw data values and sort them in ascending numerical order from smallest to largest.



2 Apply Specific Metric Algorithm

Execute the precise rule for the desired statistical summary: sum and divide for mean, locate center for median, count frequencies for mode, or subtract extrema for range.



Data Set Size (n)



Ordered List Verification



Which measure of central tendency best represents skewed data with extreme outliers?

Data contains extreme high or low values (skewed distribution)

→ Select the median, as it locates the midpoint without being pulled by distant outliers.

Data is symmetric with no extreme values

→ Select the mean, as it utilizes all numerical values in the calculation.



Calculating the Mean

The arithmetic mean combines every data point into an aggregate sum and distributes it evenly across the total number of entries.

- Add every number in the data set together to find the numerator.
- Count the total number of entries (n) to find the denominator.
- Divide the sum by n to compute the mean value.



Common Mistake: Skimming Zeros

Values of zero must be counted in the total item count (n) even though adding zero does not increase the sum.



1 Sum All Observations

Given data {56, 69, 82, 87, 87, 93},
sum: $56 + 69 + 82 + 87 + 87 + 93 = 474$.



2 Divide by Count

Count $n = 6$. Compute $474 / 6 = 79$.
The mean is 79.

Sum of Values ($\sum x$)

Total Observations (n)

Calculated Mean ($\bar{x}$)



Does the data set contain negative numbers?

Negative numbers are present

→ Preserve negative signs during addition so they subtract from the cumulative sum before dividing.

Only positive numbers are present

→ Sum standard positive integers directly and divide by the count.

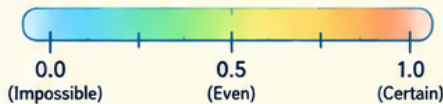
Probability Fundamentals Mind Map



Probability Scale & Core Formula

Probability quantifies the likelihood of an event occurring on a numerical scale from 0 to 1 (or 0% to 100%). A probability of 0 signifies an impossible event, 0.5 indicates equally likely outcomes, and 1 denotes a certain event. For any event with equally likely outcomes, theoretical probability is calculated by dividing favorable outcomes by total possible outcomes.

- $P(\text{Event}) = (\text{Number of favorable outcomes}) / (\text{Total number of possible outcomes})$
- Range constraint: $0 \leq P(E) \leq 1$; sum of all complementary elementary events equals 1
- Complementary rule: $P(\text{not } E) = 1 - P(E)$



$$P(E) = \frac{n(E)}{n(S)}$$



Scale Reference Points

- 0.0 = Impossible (rolling a 7 on a standard 6-sided die);
- 0.5 = Even chance (flipping heads on a fair coin);
- 1.0 = Certain (rolling a number less than 7 on a standard die).



1 Identify Total Outcomes

List the full sample space S to count every distinct, equally likely outcome $n(S)$.



2 Count Favorable Outcomes

Count the subset of outcomes that satisfy condition E to find $n(E)$.



3 Compute and Simplify

Calculate $P(E) = n(E) / n(S)$ and simplify the fraction to lowest terms or express as a decimal.



Are all individual outcomes in the sample space equally likely?



Yes, outcomes are equally likely (e.g., fair coin, unbiased die).

Use the standard uniform formula $P(E) = n(E) / n(S)$.



No, outcomes have uneven likelihoods (e.g., weighted spinner, biased coin)

Sum the specific probabilities assigned to each distinct favorable outcome rather than counting raw frequencies.

Probability Fundamentals



Sample Spaces & Tree Diagrams

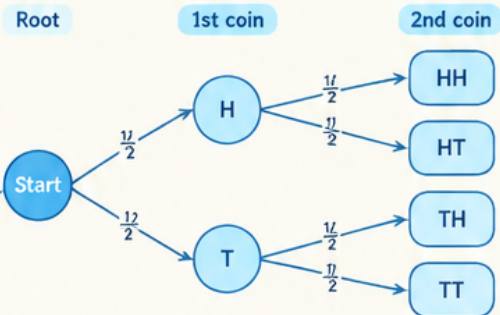
The sample space is the complete set of all possible outcomes of an experiment. When an experiment involves multiple stages or compound trials (such as flipping two coins or rolling two dice), structured representations such as organized lists, grids, or tree diagrams prevent omitted outcomes.

- Sample space notation: $S = (\text{outcome 1, outcome 2, ..., outcome } n)$
- Two-stage grid: Rows represent stage 1 outcomes, columns represent stage 2 outcomes
- Tree diagrams: Each branch represents an outcome, with path branch multiplication for compound probabilities



Tree Diagram Rule

Multiply probabilities along a single path to find the joint probability of consecutive events; add the resulting products across separate paths to find combined event probabilities.



1 Draw Primary Branches

Branch out from the starting node for each possible first-stage outcome.



2 Draw Secondary Branches

From the tip of each first-stage branch, add sub-branches for all possible second-stage outcomes.



3 Enumerate the Sample Space

Trace each path from root to final leaf to write the full set of compound outcomes.



Theoretical vs. Experimental Probability & Common Pitfalls

Theoretical probability predicts what should happen over an infinite number of identical trials based on mathematical symmetry. Experimental probability (relative frequency) records what actually happened across a finite set of observed trials: $\text{Relative Frequency} = (\text{Observed favorable trials}) / (\text{Total completed trials})$. Under the Law of Large Numbers, experimental relative frequency approaches theoretical probability as the number of trials increases.

- Theoretical: $P(\text{Heads}) = 1/2 = 0.5$ based on 2 symmetrical faces
- Experimental: Flipping a coin 20 times and getting 13 Heads gives an experimental probability of $13/20 = 0.65$
- Convergence: Running 1,000 or 10,000 trials narrows the gap between observed relative frequency and theoretical odds



Common Mistake: The Gambler's Fallacy

Believing that past independent trials influence future trials (e.g., "A coin landed on Tails 4 times in a row, so Heads is due on the next flip"). For fair, independent events, $P(\text{Heads})$ remains exactly $1/2$ on every single toss.



Are you calculating expected mathematical odds or evaluating laboratory/trial data?



Expected mathematical odds prior to trial execution

Compute theoretical probability $P(E) = n(E) / n(S)$.



Observed empirical trial data from an experiment

Compute experimental probability by dividing tally counts of observed events by total conducted trials.



Worked Numeric Examples

Worked numeric demonstrations verifying probability calculations for single-stage and compound sample spaces using standard arithmetic simplification.



Single Die Example

Rolling an even number on a standard 6-sided die:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Favorable (even) = $\{2, 4, 6\}$

$$P(\text{Even}) = \frac{3}{6} = \frac{1}{2} = 0.5 \quad (50\%)$$



Two-Coin Toss Example

Flipping two fair coins:

$$S = \{HH, HT, TH, TT\}$$

Favorable (at least one H) = $\{HH, HT, TH\}$

$$P(\text{At least one H}) = \frac{3}{4} = 0.75 \quad (75\%)$$



Card Draw Example

Drawing an Ace from a standard 52-card deck:

Total outcomes = 52

Favorable (Aces) = 4

$$P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$$



Simplification Check

Always reduce fractional probabilities using greatest common factors (GCF): $3/6$ divides by 3 to reach $1/2$; $4/52$ divides by 4 to reach $1/13$.